Semiclassical monopole calculations in supersymmetric gauge theories

Davies, N Michael

Semiclassical monopole calculations in supersymmetric gauge theories

N. Michael Davies

A thesis submitted for the degree of Doctor of Philosophy

University of Durham

Department of Physics

2000

Abstract

We investigate semiclassical contributions to correlation functions in $\mathcal{N}=1$ supersymmetric gauge theories. Our principal example is the gluino condensate, which signals the breaking of chiral symmetry, and should be exactly calculable, according to a supersymmetric non-renormalisation theorem. However, the two calculational approaches previously employed, SCI and WCI methods, yield different values of the gluino condensate.

We describe work undertaken to resolve this discrepancy, involving a new type of calculation in which the space is changed from $\mathbb{R}^4$ to the cylinder $\mathbb{R}^3 \times S^1$. This brings control over the coupling, and supersymmetry ensures that we are able to continue to large radii and extract answers relevant to $\mathbb{R}^4$. The dominant semiclassical configurations on the cylinder are all possible combinations of various types of fundamental monopoles. One specific combination is a periodic instanton, so monopoles are the analogue of the instanton partons that have been conjectured to be important at strong coupling. Other combinations provide significant contributions that are neglected in the SCI approach.

Monopoles are shown to generate a superpotential that determines the quantum vacuum, where the theory is confining. The gluino condensate is calculated by summing the direct contributions from all fundamental monopoles. It is found to be in agreement with the WCI result for any classical gauge group, whereas the values for the exceptional groups have not been calculated before. The ADS superpotential, which describes the low energy dynamics of matter in a supersymmetric gauge theory, is derived using monopoles for all cases where instantons do not contribute. We report on progress made towards a two monopole calculation, in an attempt to quantify the missed contributions of the SCI method. Unfortunately, this eventually proved too complicated to be feasible.

Acknowledgements

First and foremost, I would like to thank my supervisor, Valya Khoze, for a great deal of excellent technical advice, as well as energy and enthusiasm, over these last three (and a bit) years. It goes without saying that this thesis would never have existed without him. I would also like to acknowledge the useful contributions of our collaborators, Dr. T. J. Hollowood and Dr. M. P. Mattis.

A special mention should be made for Prof. Mike Pennington, who has been head of the physics department for most of my time here, but more importantly was always of great help with any problem I brought to him, even after I told him I didn't want to work on his research topic!

Thanks to Les Steele for the entertaining Tuesday afternoons at Durham Sixth Form Centre, and to the staff of the PRI for organising the Researcher-in-Residence scheme.

Many thanks go to my long-suffering office-mates, Anja, Jeppe and Mr. Byte, for sharing all the fun! In particular, I must thank Anja for looking after me on several occasions; how I've survived the last few months without fruit supplements I don't know! And how I would have written up without the help of Jeppe's expert computer knowledge is another mystery ... Thanks also to all the other members of the CPT for much socialising: and especially to Stu, for witnessing my attempts to learn Italian without laughing too much; and to Linda, for telling me a fact that proved to be the inspiration for some new adjoint fermion zero modes, but more importantly for making sure I practised guitar from time to time.

During my time in Durham I have been kept sane by frequent stress-relief with the Karate-do-Shotokai club. I would like to thank my instructors, Colin, Minh and Pete, for their excellent tuition. I am also hugely grateful to Rachel S. for always being cool, never thinking that I'm being too decadent, and for the nicest thesis proof-reading anyone's ever had!

Thanks to all those members of Palatine house in my first year, for the friendly and welcoming environment that was a great start to my PhD! There are too many of you to mention, but I would like to say grazie mille to Charlotte, Michelle and Eeny-Meeny for holidays in Italy, being there and understanding me, and everything!

Meanwhile, I was living in a big red funny-looking house at 115 Gilesgate, which was made as fun as it was by my fantastic housemates: Ania, Berni, Fatih, Gavin, Matt, Nina and Richard. Cheers to Matt for the jamming sessions, and to Nina for all her energy and the only poem anyone's ever written for me! Special thanks go to Gavin for his support and friendship through it all, and for never thinking it was boring to talk to me about music, or comics, or even maths or just how knackered we were ... I am also indebted to Ania, for her survival tips about Poland that may just come in useful again, but mostly for helping me during all the darkest moments of the last few years; maybe it was just your bad luck to be around at those points, but your advice was always brilliantly effective.

I am very grateful to those who helped me out during a recent spell of "homelessness": Gavin & Clare, Ania, Kate, Tom & Cathy-Ann, Jeppe and John & Anne-Sophie. I don't know how I'd have got to this point without you!

In no particular order, many thanks to Andi and Ewelina for the holiday in Poland, to Corr for the trip to NY, and to Chris and Carrie for all the parties! I am forever grateful to Feđa, just for being fabulous, but also because I wouldn't have even started in Durham if it wasn't for him.

Naturally I wish to thank my family, Mum, Dad and Nicola, for far more than I can describe in this small space. Suffice it to say: I love you all! And thanks for finding me a job, Mum!

During my research I was funded by a PPARC studentship.

Declaration

I declare that no material presented in this thesis has previously been submitted for a degree at this or any other university.

The research described in this thesis has been carried out in collaboration with Dr. V. V. Khoze, Dr. T. J. Hollowood and Dr. M. P. Mattis and has been published as follows:

[1] N. Michael Davies, Timothy J. Hollowood, Valentin V. Khoze and Michael P. Mattis, *Gluino condensate and magnetic monopoles in supersymmetric gluodynamics*, Nucl. Phys. **B559** (1999) 123, `hep-th/9905015`.

[2] N. Michael Davies and Valentin V. Khoze, *On Affleck-Dine-Seiberg superpotential and magnetic monopoles in supersymmetric QCD*, JHEP **01** (2000) 015, `hep-th/9911112`.

[3] N. Michael Davies, Timothy J. Hollowood, and Valentin V. Khoze, *Monopoles, affine algebras and the gluino condensate*, `hep-th/0006011`.

The main contributions of the author are contained in section 6.1 [3], and chapter 7 [unpublished].

Statement of copyright

Contents

List of Tables

Chapter 1

Introduction

The mathematical foundations of particle physics are quantum field theories with local gauge invariance. The standard first step towards calculations in these theories is to make a perturbative expansion in powers of the dimensionless coupling constant, g, which must be small for this approach to be valid. Perturbative calculations have been used to make predictions that have not significantly disagreed with experiment for over twenty years. Nevertheless, there are many non-perturbative phenomena, chiefly those associated with low energy hadronic physics where the coupling is large, for which we have a much poorer understanding.

In all four dimensional non-abelian gauge theories, it is known that, even for small values of the coupling constant, there are contributions that do not appear in the perturbative expansion. They arise because of the existence of non-trivial solutions of the classical equations of motion, known as *instantons*. We can make a perturbative expansion around these solutions, instead of the trivial version, in a *semiclassical calculation* that leads to results of the form

$$e^{-Cg^{-2}}\left(a_0 + a_1 g^2 + a_2 g^4 + \dots\right), \tag{1.1}$$

with constants C and $\{a_n\}$. The exponential factor has an essential singularity at $g = 0$ and so cannot be written as a Taylor series in g. For this reason, these contributions are called non-perturbative, although just as for conventional perturbative calculations, we have no way of estimating their effects when the coupling constant is large.

In principle, instantons can be used to provide analytical information about various aspects of the low energy behaviour of theories, but due to the uncontrollably large coupling in that situation, semiclassical calculations normally give infrared divergent answers that are difficult to interpret. In the last few years, however, well-defined results have been obtained using instantons in *supersymmetric* theories, that is, theories possessing invariance under transformations that mix bosonic and fermionic degrees of freedom. Supersymmetry has attracted a great deal of interest, because it appears ubiquitously in viable generalisations of the standard model, the set of gauge theories that describe well our current knowledge of particle physics, and because it leads to the simplification of many issues through the use of powerful analytical tools. For example, symmetry allows the proof of a non-renormalisation theorem, which states that exact results for certain quantities may be found using semiclassical calculations. This was first achieved for the larger, extended supersymmetries, with $\mathcal{N} = 2$ or $\mathcal{N} = 4$, where the power of extra symmetry also gives greater control over the size of the coupling.

In contrast, for theories with the minimal $\mathcal{N} = 1$ supersymmetry, there are inconsistencies between different instanton calculations of a fundamentally important correlation function, the gluino condensate, that have not been explained in nearly fifteen years! The two types of approach are SCI (strong coupling instanton) calculations, which are direct but contain no attempt to ensure that the coupling is small, and WCI (weak coupling instanton) calculations where quantities are found indirectly, via a modified theory in which the size of the coupling is controlled. We believe that the SCI calculations are incorrect, because while instantons are the only non-trivial solutions that may contribute for small values of the coupling, there may be other important configurations at large coupling that are missed in the SCI approach.

In this thesis, we will describe a new type of calculation, with the aim of resolving this dispute. The strategy is to modify space from $\mathbb{R}^4$ to $\mathbb{R}^3 \times S^1$; supersymmetry will enable the large radius limit to be taken for all results, so that answers relevant to $\mathbb{R}^4$ may be obtained. This modification also has the effect of providing control over the size of the coupling, and regulating the undesirable properties of candidates for the neglected configurations in the SCI method. It therefore combines the reliability of the WCI calculations with a mechanism for understanding the shortcomings of the SCI approach. We find that on the four dimensional cylinder $\mathbb{R}^3 \times S^1$, the relevant semiclassical configurations are *monopoles*, and together with instantons, these solutions are in principle sufficient to exactly determine any calculable quantity in supersymmetric gauge theories on $\mathbb{R}^4$.

We resume in chapter 2 with a general introduction to semiclassical calculations, and a description of the most widely used semiclassical configurations, Yang-Mills instantons. More required knowledge is presented in chapter 3, where we discuss the relevant implications of supersymmetry. We also define the gluino condensate and review previous methods used to calculate it.

In chapter 4, we describe the essence of the calculations on $\mathbb{R}^3 \times S^1$ presented in this thesis, and explore the properties of the dominant semiclassical configurations, monopoles, on that space. The calculational method is investigated in detail in chapter 5, where we apply it to the determination of the gluino condensate with gauge group $SU(2)$, and to the calculation of the superpotential in the low energy effective action of the theory on $\mathbb{R}^3 \times S^1$ [1].

Chapters 6 and 7 contain various attempts to generalise the one monopole calculations of chapter 5. First we present one monopole calculations of the gluino condensate for any gauge group [3], and of the ADS superpotential that describes the low energy dynamics of matter included in a supersymmetric gauge theory [2]. In the later chapter we report on the progress made towards a two monopole calculation designed to check the monopole hypothesis, by explicitly evaluating the contributions missed by the SCI method, in the case of gauge group $SU(2)$. The Nahm construction of multi-monopole solutions is discussed, and investigated in detail in the one monopole sector. An identity fulfilled by a Green's function that appears in this formalism is obtained, and this enables the identification of two previously unknown adjoint fermion zero modes. They are possessed by every monopole configuration with winding number greater than one, and are associated with a natural symmetry of the Nahm construction. These results are only a small part of the technology necessary to attempt this calculation, however, and unfortunately it proved to be impossible.

Finally, we will draw our conclusions in chapter 8.

Chapter 2

Semiclassical physics and instantons

We shall begin by describing the methods and introducing the necessary terminology for the calculations presented in this thesis. We do this partly by using the example of Yang-Mills instantons, in the second section of this chapter. Good references for this background material can be found in textbooks and reviews [4, 5, 6] and in the seminal paper by 't Hooft [7].

2.1 The formalism of semiclassical calculations

2.1.1 Instantons

For now we shall consider a general bosonic field ϕ, which may be thought of as a real scalar field, in order to reduce complications such as having to follow index structures, and so as to show most clearly the general features of a semiclassical calculation. Then we will add details and improvements until the formalism is sophisticated enough to cover all cases, including gauge fields and fermionic fields.

In the Euclidean version of the theory, which is the case with the greatest security in issues of convergence, the partition function has the form

$$Z = \int d\phi \, e^{-S[\phi]}, \tag{2.1}$$

and quantities of interest to calculate are the correlation functions, or multi-point functions,

$$\left\langle \prod_n \phi(x_n) \right\rangle = \left\langle 0 \left| \prod_n \phi(x_n) \right| 0 \right\rangle = Z^{-1} \int d\phi \, e^{-S[\phi]} \prod_n \phi(x_n). \tag{2.2}$$

The action S is the integral over the infinite volume of Euclidean space of a function of ϕ, and so unless the field is in a special configuration the action will not be finite. The functional integrals above both contain e^{-S} in the integrand, so we may expect that they only have non-zero contributions from regions in field space where ϕ obeys the appropriate conditions to ensure $S < \infty$. More precisely, the heavy suppression of large values of the action by the factor e^{-S} means that we can reliably approximate such an integral by using expansions around every finite minimum of the action to approximate all the significant contributions to the integral. This is the essence of semiclassical calculations.

Let ϕ_{class} be a field configuration that locally minimises the action (which implies it is a solution of the classical Euler-Lagrange equations of motion), and where the value of the minimum is finite;

$$\left. \frac{\delta S}{\delta \phi} \right|_{\phi = \phi_{\text{class}}} = 0, \quad S[\phi_{\text{class}}] < \infty. \tag{2.3}$$

Solutions of this type are called *instantons*, and it is the neighbourhoods of these points in field space that dominate the partition function and correlation functions.

2.1.2 The semiclassical approximation

In order to approximate contributions to the functional integrals due to the instanton ϕ_{class}, we first define $\delta\phi$ as the fluctuation of the quantum field ϕ from the classical solution,

$$\phi = \phi_{\text{class}} + \delta\phi, \tag{2.4}$$

and then expand the action about ϕ_{class},

$$\begin{aligned} S[\phi] = S[\phi_{\text{class}}] &+ \int d^d x \, \delta\phi(x) \left. \frac{\delta S}{\delta \phi(x)} \right|_{\phi = \phi_{\text{class}}} \\ &+ \frac{1}{2} \int d^d x \, d^d y \, \delta\phi(x) \left. \frac{\delta^2 S}{\delta\phi(x) \delta\phi(y)} \right|_{\phi = \phi_{\text{class}}} \delta\phi(y) + \dots . \end{aligned} \tag{2.5}$$

From the definition of ϕ_{class}, it is obvious that the term linear in $\delta\phi$ vanishes. Furthermore, we shall not retain terms in the expansion past quadratic order in this thesis, a truncation we will call the semiclassical approximation, and which we discuss in more detail below. In this case, the functional integrals have Gaussian form and can be evaluated.

To see this, consider

$$\Sigma_{\text{class}}(x,y) = \left.\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\right|_{\phi=\phi_{\text{class}}}, \tag{2.6}$$

which is a real symmetric operator and therefore has a complete set of eigenfunctions with real eigenvalues, $\sigma_i \in \mathbb{R}$,

$$\int d^d y\, \Sigma_{\text{class}}(x,y) f_i(y) = \sigma_i f_i(x), \tag{2.7}$$

(no implicit summation over i is intended), where we can choose the functions $\{f_i\}$ to be orthonormal,

$$\int d^d x f_i(x) f_j(x) = \delta_{ij}. \tag{2.8}$$

Expressing $\delta\phi$ in terms of these eigenfunctions,

$$\delta\phi = \sum_i c_i f_i, \tag{2.9}$$

we find, to quadratic order in $\delta\phi$,

$$e^{-S[\phi]} = e^{-S[\phi_{\text{class}}]} e^{-\frac{1}{2}\sum_i \sigma_i c_i^2}, \tag{2.10}$$

which is an exponentially decaying Gaussian factor because we know $\sigma_i \geqslant 0$ as ϕ_{class} minimises the action, so $\Sigma_{\text{class}}(x,y)$ is a positive semi-definite operator.

Up to this point we have not explicitly defined the measure in the functional integrals, but we have now introduced the convenient variables to use in the neighbourhood of ϕ_{class}. First, we can work with the quantum fluctuation $\delta\phi$ rather than the full field ϕ, and then we may parametrise the degrees of freedom of $\delta\phi$ as the coefficients $\{c_i\}$ in equation (2.9), so

$$\int d\phi = \int d\delta\phi = N \int \prod_i \frac{dc_i}{\sqrt{2\pi}}, \tag{2.11}$$

including a constant N to account for our ignorance of the overall normalisation. Then, for example, we can find the contribution to the partition function due to ϕ_{class}, in the semiclassical approximation,

$$Z_I = Ne^{-S[\phi_{\text{class}}]}\prod_i \int \frac{dc_i}{\sqrt{2\pi}} e^{-\frac{1}{2}\sigma_i c_i^2} \tag{2.12}$$

$$= Ne^{-S[\phi_{\text{class}}]}\prod_i \sigma_i^{-\frac{1}{2}} \tag{2.13}$$

$$= Ne^{-S[\phi_{\text{class}}]}\det{}^{-\frac{1}{2}}\Sigma_{\text{class}}, \tag{2.14}$$

where we have assumed that all of the eigenvalues are non-zero, $\sigma_i > 0$. How to deal with the case that some of the eigenvalues vanish, as occurs in any interesting physical model, will be discussed in section 2.1.3. Even if the eigenvalues are all positive, however, the determinant above still needs careful definition. The spectrum of Σ_{class} is generally continuous, and the product cannot be taken over an uncountable set of numbers. This situation may be avoided by imposing periodic boundary conditions, which makes the spectrum discrete. Still, though, the spectrum is unbounded from above and so the product is formally divergent. This requires the constant N to be specifically chosen to normalise the determinant in an appropriate way, as we shall discuss at the end of this section. Finally, in quantum field theories the normalised determinant contains divergences that must be removed by regularisation and renormalisation, which will be considered in section 2.1.6.

To calculate the contribution due to the instanton to the correlation functions to the same order in $\delta\phi$, we need only include the zeroth order values of the field insertions, namely $\phi_{\text{class}}(x_n)$, hence

$$Z\left\langle \prod_n \phi(x_n)\right\rangle = e^{-S[\phi_{\text{class}}]}\int d\delta\phi \,\exp\left(-\frac{1}{2}\int \delta\phi\Sigma_{\text{class}}\delta\phi\right)\prod_n \phi_{\text{class}}(x_n). \tag{2.15}$$

Before we start refining this method to improve its range of applicability, let us discuss in greater depth the nature of the semiclassical approximation. We shall be concerned below with gauge theories involving a coupling g, and in which the fields may be scaled such that the coupling appears in the action only as a prefactor g^{-2}; to make this explicit we could send $S \to g^{-2}S$ above. The corrections to a semiclassical approximation result may be arranged in powers of g, leading to generic answers of the form

$$e^{-g^{-2}S[\phi_{\rm class}]}\left(a_0 + a_1 g^2 + a_2 g^4 + \dots\right), \tag{2.16}$$

with constants $\{a_n\}$ and where a_0 gives the level of the semiclassical approximation. This includes a series reminiscent of a perturbation theory result, and indeed our expansion corresponds to considering perturbations around the instanton. The classical solution may take a trivial form, ϕ being equal to a constant, with vanishing action, and this case is in fact conventional perturbation theory. However, if there are non-trivial minima of the action, we can be sure that their contributions are not reproduced by conventional perturbation theory, because the function $e^{-g^{-2}S[\phi_{\rm class}]}$ with $S[\phi_{\rm class}] \neq 0$ cannot be written as a Taylor series in g about zero, it is non-analytic at that point. Therefore, semiclassical calculations about non-trivial solutions represent non-perturbative information. Nevertheless, for these calculations to be well defined, the series in equation (2.16) must converge, and a necessary condition is that the coupling g be less than one, just as in conventional perturbation theory[1]. Note that if we retained $\hbar$ as a dimensionful constant instead of setting it to one, the functional integrals would contain $e^{-\hbar^{-1}S}$ and so $\hbar$ would shadow g^2 in the above discussion. Then the convergence requirement would be that $g^2\hbar$ should be small, which is where the name semiclassical originates from. Also, counting $g^2\hbar$ factors shows that each subsequent term in equation (2.16) represents perturbative contributions including one more loop, and the semiclassical approximation is equivalent to restricting the perturbation theory around the instanton to one loop.

We can use the recognition that conventional perturbation theory is a special case of a semiclassical calculation, about a trivial classical solution, to remove the unknown factor N from equation (2.14). If the perturbation theory contribution to the partition function is

$$Z_0 = N\det{}^{-\frac{1}{2}}\Sigma_0, \tag{2.17}$$

then we can consider the ratio of Z_I to Z_0 and find

$$Z_I = Z_0 e^{-S[\phi_{\rm class}]}\frac{\det^{-\frac{1}{2}}\Sigma_{\rm class}}{\det^{-\frac{1}{2}}\Sigma_0}. \tag{2.18}$$

[1]In fact the series will be *asymptotic*, and should be approximated by truncating it after the smallest term; but we still require the coupling to be small in order that the terms initially decrease.

2.1.3 Bosonic zero modes

The expression (2.13) is divergent if any of the eigenvalues vanishes, which is not surprising as in that case the relevant exponential factors are unity instead of being decaying functions that help the integrals converge. Furthermore, any symmetry of the system implies a zero eigenvalue, because there must be a corresponding direction in field space along which the action remains the same. All theories of interest here possess at least Lorentz symmetry and local gauge invariance, if not supersymmetry, so we must have a prescription for coping with the zero eigenvalue eigenmodes of the bosonic operator Σ_{class}, or the bosonic zero modes.

We cannot help the divergence of instanton contributions to the partition function in purely bosonic theories, but this situation will be drastically modified upon adding fermions to the theory so that need not be of concern presently. What we might hope to achieve is to rewrite the functional integral measure so that we have a convenient expression for use in calculating correlation functions; the inclusion of various fields in the integrand may improve the convergence properties of the integral. This is just what we shall aim for here.

A useful concept is that of *collective coordinates* [8]. Every symmetry has an associated physical parameter in the classical solution, for example the position of localisation of the solution corresponding to translational symmetry, or a measure of the size of the solution for the case of symmetry under scale changes. The action of any symmetry on the classical solution can in fact be taken to be changing the value of the appropriate parameter, or collective coordinate, giving another solution with the same action.

Continuing to work up from the simplest possible case, let us investigate the case of just one symmetry, with a collective coordinate τ. Then there is a one-parameter family of classical solutions with different values of τ, $\{\phi_{\text{class}}(x;\tau)\}$. If we consider $\phi(x)$ equal to a solution infinitesimally displaced from one with the particular value τ,

$$\phi(x) = \phi_{\text{class}}(x;\tau+\delta\tau) = \phi_{\text{class}}(x;\tau) + \frac{d\phi_{\text{class}}(x;\tau)}{d\tau}\delta\tau + \dots , \tag{2.19}$$

then considering the expansion of the action as in equation (2.5), the term of order $\delta\tau^2$ shows that $\frac{d\phi_{\text{class}}}{d\tau}$ is an unnormalised zero mode of Σ_{class}. By a suitable choice of labels for the eigenfunctions and their eigenvalues, we can write $\sigma_0 = 0$ and

$$\frac{d\phi_{\text{class}}}{d\tau} = \left(\int d^d x \left(\frac{d\phi_{\text{class}}}{d\tau} \right)^2 \right)^{\frac{1}{2}} f_0. \tag{2.20}$$

This appears multiplied by $\delta\tau$ in the fluctuation $\delta\phi$, which indicates that a simplification may be to change variables in the functional integrals from the coefficient of the zero mode, c_0, to the collective coordinate τ, a more natural and convenient choice.

In order to realise this change of variables and determine the relevant Jacobian factor, we shall insert into the functional integrals a Fadeev-Popov unity operator,

$$1 = \int d\tau\, \delta\left(\left\langle \delta\phi, \dot{\phi}_{\text{class}} \right\rangle \right) \Delta\left(\phi_{\text{class}}\right), \tag{2.21}$$

with $\langle f, g\rangle = \int d^d x\, f(x) g(x)$ an obvious scalar product for functions, and $\dot{\phi}_{\text{class}} = \frac{d\phi_{\text{class}}}{d\tau}$. The normalisation Δ must be arranged to make the above relation hold. The rules of calculus show that it should take the form

$$\Delta\left(\phi_{\text{class}}\right) = \left| \frac{d}{d\tau} \left\langle \delta\phi, \dot{\phi}_{\text{class}} \right\rangle \right| \tag{2.22}$$

$$= \left| - \left\langle \dot{\phi}_{\text{class}}, \dot{\phi}_{\text{class}} \right\rangle + \left\langle \delta\phi, \ddot{\phi}_{\text{class}} \right\rangle \right|, \tag{2.23}$$

where the second term in the final expression can be neglected in the semiclassical approximation. The delta function is only non-zero for fluctuations orthogonal to the zero mode.

The instanton contribution to the partition function becomes

$$Z_I = N e^{-S[\phi_{\text{class}}]} \int \left(\prod_i \frac{dc_i}{\sqrt{2\pi}} \right) \int d\tau\, e^{-\frac{1}{2}\sigma_i c_i^2} \left\langle \dot{\phi}_{\text{class}}, \dot{\phi}_{\text{class}} \right\rangle \delta\left(\left\langle \delta\phi, \dot{\phi}_{\text{class}} \right\rangle \right). \tag{2.24}$$

Recall that $\delta\phi = \sum_i c_i f_i$, and $\dot{\phi}_{\text{class}} = \left\langle \dot{\phi}_{\text{class}}, \dot{\phi}_{\text{class}} \right\rangle^{\frac{1}{2}} f_0$, so we have

$$\delta\left(\left\langle \delta\phi, \dot{\phi}_{\text{class}} \right\rangle \right) = \delta\left(\left\langle \dot{\phi}_{\text{class}}, \dot{\phi}_{\text{class}} \right\rangle^{\frac{1}{2}} c_0 \right) \tag{2.25}$$

$$= \left\langle \dot{\phi}_{\text{class}}, \dot{\phi}_{\text{class}} \right\rangle^{-\frac{1}{2}} \delta\left(c_0\right). \tag{2.26}$$

This allows us to perform the c_0 integration and find

$$Z_I = N e^{-S[\phi_{\text{class}}]} \prod_{i \neq 0} \int \frac{dc_i}{\sqrt{2\pi}} e^{-\frac{1}{2}\sigma_i c_i^2} \int \frac{d\tau}{\sqrt{2\pi}} \left\langle \dot{\phi}_{\text{class}}, \dot{\phi}_{\text{class}} \right\rangle^{\frac{1}{2}} \tag{2.27}$$

$$= N e^{-S[\phi_{\text{class}}]} \det{}'^{-\frac{1}{2}} \Sigma_{\text{class}} \int \frac{d\tau}{\sqrt{2\pi}} \left\langle \dot{\phi}_{\text{class}}, \dot{\phi}_{\text{class}} \right\rangle^{\frac{1}{2}}, \tag{2.28}$$

where det$'$ is the determinant of the operator restricted to the non-zero eigenvalue subspace, that is not including the zero eigenvalues. The instanton contribution to the partition function is still divergent because it contains an integral over an unrestricted variable, but it gives us the correctly normalised instanton measure. Once we have cured the divergence of the partition function by including fermions, this will be an essential element in the calculation of correlation functions.

This result can be easily generalised to find the instanton measure with n_B bosonic zero modes,

$$Z_I = Ne^{-S[\phi_{\text{class}}]}\det{}'^{-\frac{1}{2}}\Sigma_{\text{class}} \int \prod_{i=1}^{n_B} \frac{d\tau_i}{\sqrt{2\pi}} \det \left\langle \frac{\partial}{\partial\tau_j}\phi_{\text{class}}, \frac{\partial}{\partial\tau_k}\phi_{\text{class}} \right\rangle^{\frac{1}{2}}. \tag{2.29}$$

which includes the Jacobian factor

$$\mathcal{J}_B = \left(\frac{1}{\sqrt{2\pi}}\right)^{n_B} \det \left\langle \frac{\partial}{\partial\tau_j}\phi_{\text{class}}, \frac{\partial}{\partial\tau_k}\phi_{\text{class}} \right\rangle^{\frac{1}{2}}. \tag{2.30}$$

2.1.4 Gauge fields

Field theories with local gauge symmetry are the most interesting and important cases in particle physics. All of the theories that together form the phenomenally successful standard model of particle physics are gauge theories. In this thesis we shall be concerned with semiclassical calculations in such theories, and here we will discuss the necessary modifications to the formalism, given above only for one scalar field, to extend it to apply to theories with a gauge field.

The solutions to the equations of motion with finite action in pure Yang-Mills theory in four dimensional Euclidean space are the archetypal configurations for semiclassical calculations. They are known as Yang-Mills instantons and their properties are discussed in section 2.2

A gauge field v_m can be viewed as four scalar fields, linked together as a vector with regard to Lorentz transformations, and with some physical redundancy due to many different values of the field falling into the same gauge equivalence class. The first point here doesn't lead to significant changes in the semiclassical method; the generalisation from one to many scalar fields is trivial, and the fact that they are labelled by a vector index is never important as the indices arrange into a covariant expression at every stage. The last property, however, requires some attention. Gauge transformations of the classical solution do not change the action and this suggests the existence of an infinite number of bosonic zero modes, since the group corresponding to local gauge symmetry has infinite dimension. To prevent this, we should fix the gauge. Making an expansion around the instanton, we have

$$v_m = v_m^{\text{class}} + \delta v_m, \tag{2.31}$$

and the gauge of v_m^{class} is irrelevant to the calculation of any physical quantity, so we may pick a useful gauge condition for δv_m since this is the fundamental variable of the semiclassical calculation. A convenient choice of gauge is given by

$$D_m^{\text{class}}(\delta v^m) = 0, \tag{2.32}$$

where D_m^{class} is the covariant derivative with the instanton gauge field v_m^{class}. This is known as the covariant background gauge, and it forces the quantum fluctuations δv_m to be along directions in field space orthogonal to those given by gauge transformations, as shown by the following argument. If equation (2.32) holds everywhere then

$$\int d^d x \operatorname{Tr}\left[\Lambda D_m^{\text{class}}(\delta v^m)\right] = 0, \tag{2.33}$$

for any function $\Lambda(x)$ valued in the Lie algebra (or adjoint representation space) of the gauge group. We can integrate this by parts, giving

$$\int d^d x \operatorname{Tr}\left[\delta v^m D_m^{\text{class}} \Lambda\right] = 0, \tag{2.34}$$

and then, recognising that an infinitesimal gauge transformation of the classical solution is $v^{\text{class}} \mapsto v^{\text{class}} + D_m^{\text{class}}\Lambda$, we can see that equation (2.34) is the required orthogonality condition.

Once the gauge has been fixed in this way, the semiclassical calculations proceed essentially as before[2] allowing us to find, for example, the Jacobian factor for the bosonic measure. Investigations of exactly this kind, for a case of interest, will be described in appendix C. However, the partition function will still be infinite due to the bosonic zero modes of the instanton, and this causes a significant and fundamental barrier to any semiclassical calculations. The situation is aided by the inclusion of fermions in the theory, so we move to consider these next.

[2]We are neglecting here the influence of the ghosts arising from the gauge fixing procedure, but since they are described by an anticommuting field, we shall consider this after discussing the similar but more relevant case of fermionic fields.

2.1.5 Fermions and fermionic zero modes

We shall now consider a field theory containing a fermion ψ, which transforms under a spinor representation of the Lorentz group and is composed of Grassmannian or anticommuting numbers. This latter, defining characteristic of fermions is of crucial importance to their properties in semiclassical calculations, so we shall briefly review some useful formulae relevant to such variables.

Let ζ and η be Grassmannian numbers, obeying

$$\zeta\eta + \eta\zeta = 0, \quad \zeta^2 = \eta^2 = 0. \tag{2.35}$$

The rules of integration for these numbers, given by Berezin [9], are that

$$\int d\eta \, 1 = 0, \quad \int d\eta \, \eta = 1. \tag{2.36}$$

Therefore, because $\exp(\zeta A \eta) = 1 + \zeta A \eta$, we have

$$\int d\eta \, d\zeta \, \exp(\zeta A \eta) = A. \tag{2.37}$$

The measure $d\eta$ of a Grassmannian number behaves in almost the opposite manner to an integration measure from real calculus. This is most clearly demonstrated by noting from the rules above that $\int d\eta$ acts in the same way as the derivative operator $\frac{\partial}{\partial\eta}$. In fact $d\eta$ should be considered as a Grassmannian number independent of η, but defined by the requirements in equations (2.36). The order of integration is therefore important, since

$$d\zeta \, d\eta = -d\eta \, d\zeta. \tag{2.38}$$

Also, under a change of scale, say $\eta \to \zeta = a\eta$, preservation of the rules (2.36) leads to the measure transforming in the reciprocal manner,

$$d\eta \to d\zeta = \frac{d\eta}{a}. \tag{2.39}$$

We will assume that the action takes the form

$$S\left[\phi, \psi, \psi^\dagger\right] = S_B[\phi] + \int d^d x \, \psi^\dagger \mathcal{D}(\phi) \psi, \tag{2.40}$$

with some Hermitian operator $\mathcal{D}$. This is quadratic in fermionic fields without the need for any approximation. Here ϕ can stand for many bosonic fields, for example a gauge field v_m after gauge fixing. The partition function is

$$Z = \int d\phi\, d\psi\, d\psi^\dagger\, e^{-S_B[\phi]} \exp\left(-\int d^dx\, \psi^\dagger \mathcal{D}(\phi)\psi\right). \tag{2.41}$$

We shall treat the factor $e^{-S_B[\phi]}$ as before, expanding around a non-trivial classical solution and neglecting all terms of cubic or higher order in the quantum fluctuations. To the level of the semiclassical approximation, we can replace $\mathcal{D}(\phi)$ with $\mathcal{D}(\phi_{\text{class}}) = \mathcal{D}_{\text{class}}$.

We can deal with the fermions using a similar method to the bosonic case. First, we expand ψ in terms of the eigenfunctions of $\mathcal{D}_{\text{class}}$, including Grassmannian coefficients $\{\eta_i\}$,

$$\psi = \sum_i \eta_i g_i. \tag{2.42}$$

The functions $\{g_i\}$ obey

$$\mathcal{D}_{\text{class}} g_i = \delta_i g_i, \tag{2.43}$$

(with no summation over i), and an orthonormality condition

$$\int d^dx\, g_i^\dagger g_j = \delta_{ij}, \tag{2.44}$$

Then, defining the measure as $d\psi\, d\psi^\dagger = \prod_i d\eta_i\, d\eta_i^\dagger$, equation (2.37) shows that we have

$$Z_I = \int d\phi\, e^{-S_B[\phi]} \prod_i \delta_i \tag{2.45}$$

$$= \int d\phi\, e^{-S_B[\phi]} \det \mathcal{D}_{\text{class}}. \tag{2.46}$$

Now we can notice that if any of the eigenvalues of the operator $\mathcal{D}_{\text{class}}$ are zero, the instanton contribution to the partition function does not diverge, but vanishes! A normalisable solution of the equation

$$\mathcal{D}_{\text{class}} g = 0, \tag{2.47}$$

is called a fermionic zero mode. If such an eigenfunction exists, the bosonic zero modes no longer harmfully lead to an infinite partition function, but rather,

$$Z_I = 0. \tag{2.48}$$

Then, the partition function is contributed to solely by conventional perturbation theory, which at one loop level gives

$$Z = Z_0 = N \det{}^{-\frac{1}{2}} \Sigma_0 \cdot \det \mathcal{D}_0. \tag{2.49}$$

Note that Σ_0 and $\mathcal{D}_0$ are positive definite operators that do not possess zero modes, so their determinants need not be truncated to give a non-zero, finite result.

Although the semiclassical calculation of the partition function yields the value zero, there are still correlation functions to which an instanton may contribute. We shall take the case of just one normalised fermionic zero mode, g, with the associated parameter η in the expansion of ψ. Then, we can draw together the results given in equations (2.14), (2.15), (2.18), (2.29), and (2.46), and find that the semiclassical approximation value for a generic correlation function takes the form

$$\left\langle \prod_{n,p,q} \phi(x_n)\psi(x_p)\psi^\dagger(x_q) \right\rangle_I = e^{-S_B[\phi_{\text{class}}]} \frac{\det'^{-\frac{1}{2}} \Sigma_{\text{class}}}{\det^{-\frac{1}{2}} \Sigma_0} \int \prod_{i=1}^{n_B} d\tau_i \mathcal{J}_B \prod_n \phi_{\text{class}}(x_n)$$
$$\frac{\det' \mathcal{D}_{\text{class}}}{\det \mathcal{D}_0} \int d\eta \, d\eta^\dagger \prod_{p,q} \eta g(x_p) \eta^\dagger g^\dagger(x_q). \tag{2.50}$$

We have left undone the integrations over η, which is a fermionic collective coordinate, and over its complex conjugate $\eta^\dagger$. As before, $\det' \mathcal{D}_{\text{class}}$ represents the determinant of the operator $\mathcal{D}_{\text{class}}$ with zero eigenvalues excluded. This formula shows that the instanton contributions to most correlation functions are also zero. For example, any purely bosonic correlator, without insertions of ψ or $\psi^\dagger$, is proportional to $\int d\eta \, 1$, which is zero under the rules (2.36). Furthermore, if there are too many fermionic fields in the correlation function, then the anticommuting nature of η gives $\eta^2 = 0$, and this again ensures that the result vanishes. Therefore, generally, it is only in the situation that the correlation function contains exactly the same number of fermionic fields as the instanton has fermionic zero modes, that there is a non-zero value for the instanton contribution to the correlation function. This special circumstance is called *saturation* of the fermionic zero modes.

The expression above for the instanton contribution to a correlation function in a theory with a single fermion, and where the instanton has just one normalised fermionic zero mode, is easily generalised to having several fermions (all still represented by ψ, with identity labels suppressed) and n_F unnormalised fermionic zero modes, giving

$$\left\langle \prod_{n,p,q} \phi(x_n)\psi(x_p)\psi^\dagger(x_q) \right\rangle_I = e^{-S_B[\phi_{\text{class}}]} \frac{\det'^{-\frac{1}{2}}\Sigma_{\text{class}}}{\det^{-\frac{1}{2}}\Sigma_0} \int \prod_{i=1}^{n_B} d\tau_i \mathcal{J}_B \prod_n \phi_{\text{class}}(x_n)$$
$$\frac{\det'\mathcal{D}_{\text{class}}}{\det\mathcal{D}_0} \int \prod_{j=1}^{n_F} d\eta_j\, d\eta_j^\dagger\, \mathcal{J}_F \prod_{p,q} \psi_{\text{class}}(x_p)\psi^\dagger_{\text{class}}(x_q). \tag{2.51}$$

Here ψ_{class} is a linear combination of zero modes, including the fermionic collective coordinates $\{\eta_j\}$ as coefficients,

$$\psi_{\text{class}} = \sum_{j=1}^{n_F} \eta_j g_j, \tag{2.52}$$

and $\mathcal{J}_F$ is a Jacobian factor,

$$\mathcal{J}_F = \det{}^{-1} \int d^d x\, g_j^\dagger g_k \tag{2.53}$$
$$= \det{}^{-1} \int d^d x \int d\eta_j\, d\eta_k^\dagger \psi^\dagger_{\text{class}}\psi_{\text{class}}. \tag{2.54}$$

When we come to do semiclassical calculations in gauge theories, the relevant operator will be $i\overline{\sigma}^n D_n$, and we shall make great use of the Atiyah-Singer index theorem that allows us to find the number of fermionic zero modes of $i\overline{\sigma}^n D_n^{\text{class}}$, as will be discussed in section 2.2.5. However, we should already state here that if the fermions have a mass M, included in the operator as $i\overline{\sigma}^n D_n + M$, then there are *no* zero modes, so it is only in the presence of massless fermions that the problem of a divergent partition function is solved in the manner given above.

2.1.6 Regularisation and renormalisation

So far we have only considered the immediately apparent divergence of the partition function due to bosonic zero modes, which is alleviated by the inclusion of fermions in the theory. However, more subtle problems can arise because the amputated determinant $\det'\Sigma_{\text{class}}$, and its fermionic analogue $\det'\mathcal{D}_{\text{class}}$, are infinite due to the existence of many large eigenvalues. The solution to this is to regularise and renormalise the theory, just as one would in conventional perturbative calculations; these are after all ultraviolet divergences. When working with instantons it is most convenient to use the

Pauli-Villars regularisation scheme, which is motivated by imagining that, for every field in the theory, a similar field is added, but with the opposite, unphysical statistics and a large mass μ that sets the renormalisation scale. Practically, it involves replacing the determinant of any operator $\mathcal{A}$ with the ratio

$$\frac{\det \mathcal{A}}{\det \mathcal{A}^{(\mu)}}, \tag{2.55}$$

where $\mathcal{A}^{(\mu)}$ is $\mathcal{A}$ plus the appropriate power of μ. At low energies, the approximate effect is to scale the small eigenvalues by a constant; at high energies, μ is insignificant and the ratio tends to one, controlling the divergence.

We will be concerned with supersymmetric field theories in this thesis, and fortunately the Pauli-Villars scheme doesn't break supersymmetry; the extra particles would follow the physical ones and match up in supersymmetric multiplets. The dimensional reduction scheme is widely used in supersymmetric theories, indeed it is a variant of dimensional regularisation designed to preserve supersymmetry, but it is not applicable to semiclassical calculations as the instanton solution is specific to a given number of dimensions.

In the Pauli-Villars regularisation scheme, our previous expression for the instanton contribution to a correlation function, equation (2.51), becomes

$$\begin{aligned}
&\left\langle \prod_{n,p,q} \phi(x_n)\psi(x_p)\psi^\dagger(x_q) \right\rangle_I = \\
&\mu^n e^{-S_B[\phi_{\text{class}}]} \frac{\det'^{-\frac{1}{2}} \Sigma_{\text{class}}}{\det'^{-\frac{1}{2}} \Sigma^{(\mu)}_{\text{class}}} \frac{\det^{-\frac{1}{2}} \Sigma^{(\mu)}_0}{\det^{-\frac{1}{2}} \Sigma_0} \int \prod_{i=1}^{n_B} d\tau_i \mathcal{J}_B \prod_n \phi_{\text{class}}(x_n) \\
&\frac{\det \mathcal{G}_{\text{class}}}{\det \mathcal{G}^{(\mu)}_{\text{class}}} \frac{\det \mathcal{G}^{(\mu)}_0}{\det \mathcal{G}_0} \frac{\det' \mathcal{D}_{\text{class}}}{\det' \mathcal{D}^{(\mu)}_{\text{class}}} \frac{\det \mathcal{D}^{(\mu)}_0}{\det \mathcal{D}_0} \int \prod_{j=1}^{n_F} d\eta_j \, d\eta_j^\dagger \, \mathcal{J}_F \prod_{p,q} \psi_{\text{class}}(x_p)\psi^\dagger_{\text{class}}(x_q).
\end{aligned} \tag{2.56}$$

We have included the effects of ghosts, which have a quadratic operator $\mathcal{G}$ in the action ($\mathcal{G}$ is positive definite, so ghosts do not possess zero modes and their determinants do not have to be modified). Also, we have explicitly written the lowest n_B and n_F eigenvalues from $\det \Sigma^{(\mu)}_{\text{class}}$ and $\det \mathcal{D}^{(\mu)}_{\text{class}}$ respectively, in order to compare reduced determinants $\det'$, and this has given the power of μ at the front. When combined with the instanton action, this is crucial for the renormalisation group invariance (that is, μ independence) of the result. For the theories we will be discussing in this thesis,

$$\mu^n = \mu^{n_B - \frac{1}{2}n_F}. \tag{2.57}$$

The contribution from the bosonic operators is μ^{n_B}, which is easily understood. To explain why the fermionic factor is $\mu^{-\frac{1}{2}n_F}$, we first state that we will always be dealing with Weyl or chiral fermions, which couple to the quadratic operator $i\bar{\sigma}^n D_n$ in the action. This is not Hermitian, and so not immediately suitable for semiclassical analysis. Instead, we must rearrange the Weyl fermion and anti-fermion into a real Majorana fermion, then because this has half the degrees of freedom of a complex Dirac fermion, the determinant of the relevant operator appears to the power $\frac{1}{2}$, which leads directly to the fermionic contribution of $\mu^{-\frac{1}{2}n_F}$.

2.2 Yang-Mills instantons

We give here a quick overview of the features of instantons in Yang-Mills theory in four dimensional Euclidean space. These provide a good demonstration of the ideas that we have been discussing so far, as well as introducing the concept of *topological charge*, all of which will be important for the calculations presented in this thesis. When we refer to instantons in later chapters, we shall normally mean such Yang-Mills instanton configurations, as should hopefully be obvious from the context.

2.2.1 The instanton number

The action of Yang-Mills theory in the absence of matter is

$$S = \int d^4x \, \frac{1}{2g^2} \mathrm{Tr}\,(v_{mn} v^{mn}). \tag{2.58}$$

In order to have a finite action, it is clear that the field strength must tend to zero as $|x| \to \infty$. This does not imply that the gauge field has to vanish at spatial infinity, but it must have the form of a pure gauge,

$$\lim_{|x| \to \infty} v_m = iU\partial_m U^{-1}. \tag{2.59}$$

Therefore every finite action gauge configuration is associated with a mapping U from the three sphere at infinity, S^3_∞, to the gauge group. Let us start with the simplest possible case and take the gauge group to be $SU(2)$, which is also isomorphic to S^3. Therefore U maps S^3 to itself, and such mappings are topologically characterised by elements of the homotopy group

$$\pi_3(S^3) = \mathbb{Z}, \tag{2.60}$$

so we can assign to any Yang-Mills instanton an integer, known as the instanton number or Pontryagin index. This is an example of a topological charge, or winding number (referring to the winding of the hyperspheres around each other). The instanton number can be calculated using the formula

$$k = \int d^4x \, \frac{1}{16\pi^2} \mathrm{Tr}\,(v_{mn} {*v^{mn}}), \tag{2.61}$$

where ${*v_{mn}} = \frac{1}{2}\epsilon_{mnpq}v^{pq}$.

To prove this, first note that $\mathrm{Tr}\,(v_{mn} {*v^{mn}})$ is equal to the divergence of the Chern-Simons current,

$$K^m = 2\epsilon^{mnpq}\mathrm{Tr}\left(v_n\partial_p v_q - \frac{2}{3}iv_n v_p v_q\right), \tag{2.62}$$

therefore,

$$\int d^4x \, \mathrm{Tr}\,(v_{mn} {*v^{mn}}) = \int d^4x \, \partial_m K^m = \int_{S^3_\infty} (d^3x)_m \, K^m. \tag{2.63}$$

The condition of finite action, which requires $v_{mn} = 0$ on the hypersphere at infinity, implies $\epsilon^{mnpq}(\partial_p v_q - iv_p v_q) = 0$ and $v_m = iU\partial_m U^{-1}$ on S^3_∞. Using these results, the integral becomes

$$\int d^4x \, \mathrm{Tr}\,(v_{mn} {*v^{mn}}) = \int_{S^3_\infty} (d^3x)_m \, \frac{2}{3}\epsilon^{mnpq}\mathrm{Tr}\,\left(U\left(\partial_n U^{-1}\right)U\left(\partial_p U^{-1}\right)U\left(\partial_q U^{-1}\right)\right). \tag{2.64}$$

We can parametrise the above integral in terms of any three independent variables on the hypersphere, call them $\{\xi_i\}$ for $i = 1, 2, 3$. They are functions of the three coordinates $\{\theta_i\}$ of the $SU(2)$ manifold, and if we change to these as integration variables, the trace over U matrices and derivatives becomes exactly the weighting required to make the invariant measure of $SU(2)$. Therefore,

$$\int d^4x \, \mathrm{Tr}\,(v_{mn} {*v^{mn}}) \propto \mathrm{Vol}\,(SU(2)). \tag{2.65}$$

However the $\{\xi_i\}$ may map to the $\{\theta_i\}$ many times, though still continuously, so we can further write

$$\int d^4x \,\mathrm{Tr}\,(v_{mn}{*}v^{mn}) \propto k \,\mathrm{Vol}\,(SU(2))\,, \tag{2.66}$$

with k an integer giving the number of times S^3_∞ is wrapped around $SU(2)$, which we recognise as the instanton number. It remains only to find the constant of proportionality between $\int d^4x \,\mathrm{Tr}\,(v_{mn}{*}v^{mn})$ and k, and this is most easily achieved by explicitly evaluating the integral in the case where the mapping is $U_1 = \frac{1}{|x|}x_m\sigma^m$. This is a manifestly one-to-one mapping and so should have unit topological charge. It yields

$$\int_{S^3_\infty} (d^3x)_m \,\frac{2}{3}\epsilon^{mnpq}\mathrm{Tr}\,\left(U_1\left(\partial_n U_1^{-1}\right)U_1\left(\partial_p U_1^{-1}\right)U_1\left(\partial_q U_1^{-1}\right)\right) = 16\pi^2, \tag{2.67}$$

which verifies the factor shown in equation (2.61).

The situation is no more complicated for any other gauge group G, due to the result [10] that any mapping from an S^3 to G is topologically equivalent to a mapping from the S^3 to some $SU(2) \subset G$, so $\pi_3(G) = \mathbb{Z}$ and instantons are still labelled by the instanton number, given by equation (2.61).

The fact that $\mathrm{Tr}\,(v_{mn}{*}v^{mn})$ is a total derivative means that it can be added to the action without affecting the equations of motion. The extra contribution, known as the theta term, takes the form

$$S_\vartheta = -\frac{i\vartheta}{16\pi^2}\int d^4x \,\mathrm{Tr}\,(v_{mn}{*}v^{mn}), \tag{2.68}$$

and in the partition function it leads to the factor $e^{-S_\vartheta} = e^{ik\vartheta}$, so ϑ is an angular variable with period 2π.

2.2.2 The action

As we have just seen, any gauge configuration with finite action is associated with a topological charge, the instanton number. However, we are not just interested in finite action configurations, but ones that are also local minima of the action. There is no smooth deformation of a finite action gauge field that can change the topological charge[3], so we may expect to find minimal action configurations in each topological sector.

[3]The topological charge is therefore a non-dynamically conserved charge.

We seek solutions of the classical equations of motion,

$$D_m v^{mn} = 0. \tag{2.69}$$

However, we shall not attempt to solve these equations directly, but instead note that self-dual or anti-self-dual gauge fields, which obey $*v_{mn} = v_{mn}$ and $*v_{mn} = -v_{mn}$ respectively, automatically fulfil the equations of motion. This is because of the Bianchi identity,

$$D_m * v^{mn} = 0, \tag{2.70}$$

which always holds, so any gauge field for which $*v_{mn}$ is proportional to v_{mn} is a solution of equation (2.69). In Euclidean space, $** = 1$, so the only possible eigenvalues of the Hodge star operator are ± 1, and therefore it is the (anti-)self-duality conditions

$$*v_{mn} = \pm v_{mn}, \tag{2.71}$$

that are of great interest and utility. They are first order partial differential equations, simpler than the second order equations (2.69). Of course, the (anti-)self-duality equations are a sufficient but not necessary condition for the equations of motion to be obeyed, so not all classical solutions must be self-dual or anti-self-dual, but there are no interesting finite action solutions known that are not.

We can also see more directly that (anti-)self-dual configurations minimise the action, by writing[4]

$$\begin{aligned} S = &\int d^4x \frac{1}{4g^2} \mathrm{Tr}\,(v_{mn} \mp *v_{mn})(v^{mn} \mp *v^{mn}) \\ &\pm \int d^4x \frac{1}{2g^2} \mathrm{Tr}\,(v_{mn} * v^{mn}) &(2.72) \\ \geqslant &\pm \int d^4x \frac{1}{2g^2} \mathrm{Tr}\,(v_{mn} * v^{mn}). &(2.73) \end{aligned}$$

The action is also bounded by $S \geqslant 0$, so only one choice of the sign above gives a non-trivial restriction. Using the expression for the topological charge, equation (2.61), we can evaluate the bound,

$$S \geqslant \pm \frac{8\pi^2 k}{g^2}. \tag{2.74}$$

[4]It is useful to note that $*v_{mn} * v^{mn} = v_{mn} v^{mn}$ in Euclidean space.

Therefore we see that for positive k, the positive lower bound on the action is attained by taking the field to be self-dual. For negative k the gauge field must be anti-self-dual to minimise the action, and such solutions are often known as anti-instantons. In all cases the value of the action at the minimum is

$$S = \frac{8\pi^2|k|}{g^2}. \tag{2.75}$$

The method we have used here to find the instanton action is an example of the application of a Bogomol'nyi bound.

2.2.3 The BPST instanton

Yang-Mills instantons are known to exist for all values of the instanton number. The solution for $k = 1$ with gauge group $SU(2)$ was found by Belavin, Polyakov, Schwartz and Tyupkin in [11] and is known as the BPST instanton. It is

$$v_m = -2iV \frac{\sigma_{mn}\,(x^n - X^n)}{\rho^2 + |x - X|^2} V^\dagger. \tag{2.76}$$

(See appendix A for the definition of σ^{mn}.) This solution has eight parameters, which are the bosonic collective coordinates of the BPST instanton. They are:

- The four coordinates X^n of the spatial centre of the instanton, corresponding to translational symmetry;
- The three angles in the constant $SU(2)$ matrix V, which is a global gauge transformation that does not affect the covariant background gauge condition and so is a symmetry of the gauge fixed Lagrangian. Spatial rotations can be absorbed into V as shown in [12], and should not be considered to be an independent symmetry;
- The size ρ of the instanton, from dilatations or scale changes, the remaining part[5] of the conformal symmetry of pure Yang-Mills theory.

For simplicity we shall put X at the origin and set the gauge orientation V to the identity in what follows. This will not restrict the generality of the discussion, and both factors can be easily restored at all stages.

[5]Translations and rotations have already been considered. Inversions $x^2 \mapsto \frac{1}{x^2}$ are not a continuous symmetry and in fact map instantons to anti-instantons, and any special conformal transformation is equivalent to a combination of a translation and a gauge transformation, in a similar way to rotations. These issues are discussed fully in [12].

Our example of a mapping with unit winding was $U_1 = \frac{1}{|x|}x_m\sigma^m$, and for this matrix we have

$$iU_1\partial_m U_1^\dagger = -2i\frac{\sigma_{mn}x^n}{|x|^2}. \tag{2.77}$$

Therefore, the BPST instanton gauge field (with $X = 0$ and $V = 1$) can be written

$$v_m = \frac{|x|^2}{\rho^2 + |x|^2} iU_1\partial_m U_1^\dagger, \tag{2.78}$$

and as $|x|$ tends to infinity, $v_m \to iU_1\partial_m U_1^\dagger$, so this solution is clearly associated with instanton number one. Also, the field strength is

$$v_{mn} = 4i\frac{\rho^2}{(\rho^2 + |x|^2)^2}\sigma_{mn}, \tag{2.79}$$

which is self-dual by virtue of the self-duality of σ_{mn}.

Equation (2.76) is the BPST instanton written in what is known as a regular gauge, where the gauge field is non-singular everywhere. We can use the improper gauge transformation $U_1^\dagger$, which is not continuous at the origin, to move to a singular gauge. Then the gauge field has the form

$$v_m = \frac{\rho^2}{\rho^2 + |x|^2} iU_1^\dagger\partial_m U_1 = -2i\frac{\rho^2}{\rho^2 + |x|^2}\frac{\overline{\sigma}_{mn}x^n}{|x|^2}, \tag{2.80}$$

which has a pole at $x = X = 0$, the centre of the instanton configuration, but that point should be excluded since the gauge transformation was not well defined there. It also dies away much more quickly at large distances, compared to the regular gauge expression. This means that the boundary integral of K^m vanishes, but the instanton number is still unity, because on excluding the point X an additional boundary is introduced, and the contribution to k from this surface is one. The topological charge density is concentrated at the instanton centre in singular gauge. This type of situation is important to consider because for higher instanton numbers it may not be possible to find a gauge where the solution is continuous across the whole space.

The anti-instanton solution with $k = -1$ can be found from equation (2.76) by changing σ_{mn} to the anti-self-dual $\overline{\sigma}_{mn}$.

The solutions with $k = 1$ and $k = -1$ in any other gauge group are simply embeddings of the relevant $SU(2)$ instanton into an $SU(2)$ subgroup of the gauge group. They have $4c_2$ bosonic zero modes, where c_2 is the dual Coxeter number of the group, as defined in appendix B. Five of these parameters are the instanton centre and size, as above, and the remainder are angles in a constant gauge transformation that alters the embedding subgroup. This procedure gives the most general (anti-)self-dual solution with $k = \pm 1$, as shown in [13].

2.2.4 Multi-instantons

When $k = 1$ the configuration is frequently referred to as a one instanton solution, whereas for higher values of the instanton number they can be called k-instantons or multi-instantons. Self-dual multi-instantons have $4c_2k$ bosonic zero modes [13], as might be expected since a particular case of a k-instanton could be constructed starting from the approximate solution that is k well separated one instantons, each with $4c_2$ parameters, and calculating the perturbations required to make it an exact solution.

The most general self-dual multi-instanton solution was given implicitly by Atiyah, Drinfeld, Hitchin and Manin, in [14], via what is now known as the ADHM construction. We will discuss this in section 7.2.

2.2.5 Fermionic zero modes

If we include a Dirac fermion ψ in the gauge theory, then to proceed with semiclassical calculations we need to know about the eigenfunctions of the fermionic operator, and especially the zero modes, in the background of a Yang-Mills instanton. As we mentioned in section 2.1.5, only if the fermion is massless do fermionic zero modes exist. With massless fermions it is convenient to work in the Weyl representation, where ψ has the form

$$\psi = \begin{pmatrix} \psi_{L\alpha} \\ \overline{\psi}_R{}^{\dot{\alpha}} \end{pmatrix}, \tag{2.81}$$

and the gamma matrices are partitioned as

$$\gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \overline{\sigma}^m & 0 \end{pmatrix}. \tag{2.82}$$

The action written using these variables is

$$S = \int d^4x \left(\frac{1}{2g^2} \mathrm{Tr}\left(v_{mn} v^{mn}\right) + i\overline{\psi}_{\mathrm{L}} \cdot \overline{\sigma}^m D_m \psi_{\mathrm{L}} + i\psi_{\mathrm{R}} \cdot \sigma^m D_m \overline{\psi}_{\mathrm{R}} \right). \tag{2.83}$$

We are therefore interested in the zero modes of $i\overline{\sigma}^m D_m$ and $i\sigma^m D_m$. An application of the Atiyah-Singer index theorem [15] shows that the number of fermionic zero modes of $i\overline{\sigma}^m D_m$ minus the number for $i\sigma^m D_m$ is proportional to the instanton number. The constant of proportionality depends on the representation of the gauge group appropriate to ψ. For the fundamental representation, it is 1, but in this thesis we shall be more often concerned with the adjoint representation, for which the constant is $2c_2$. This result is more powerful than it may seem, because we can show that there are *no* zero modes for $i\sigma^m D_m$ if the gauge field is self-dual, so the Atiyah-Singer index theorem allows us to predict the numbers of fermionic zero modes of both operators when the gauge field is a Yang-Mills instanton.

The required proof is by contradiction. Suppose there is a normalisable solution to

$$i\sigma^n D_n \chi = 0, \tag{2.84}$$

then we can act on this equation with the operator $-i\overline{\sigma}^m D_m$ and find

$$\overline{\sigma}^m \sigma^n D_m D_n \chi = \left(\delta^{mn} + 2\overline{\sigma}^{mn}\right) D_m D_n \chi = 0, \tag{2.85}$$

using results from appendix A. The antisymmetry of $\overline{\sigma}^{mn} = -\overline{\sigma}^{nm}$ implies

$$2\overline{\sigma}^{mn} D_m D_n = \overline{\sigma}^{mn} \left[D_m, D_n\right] = -i\overline{\sigma}^{mn} v_{mn}. \tag{2.86}$$

However, $\overline{\sigma}^{mn}$ is also anti-self-dual, so if v_{mn} is self-dual then their contraction vanishes. Therefore our initial assumption leads to the conclusion

$$D^m D_m \chi = 0, \tag{2.87}$$

but $D^m D_m$ is a positive definite operator and cannot have a zero eigenvalue (in the space of normalisable, non-singular functions). An exactly similar proof shows that there are no zero modes for $i\overline{\sigma}^m D_m$ if the gauge field is anti-self-dual.

The above discussion shows, for example, that a one instanton in any gauge theory has one zero mode for each fundamental left-handed Weyl fermion in the theory [7], and $2c_2$ zero modes for each left-handed Weyl fermion that transforms according to the adjoint representation [16]; for gauge group $SU(2)$ this means four adjoint fermion zero modes. In supersymmetric pure $SU(2)$ Yang-Mills theory, the fermionic superpartner of the gauge particle transforms under the adjoint representation, and just as for the bosonic zero modes of the gauge field, we can relate its fermionic zero modes to obvious symmetries of the theory. There are two zero modes associated with supersymmetry,

$$\psi_\alpha^{\text{susy}} = \sigma^{mn}{}_\alpha{}^\beta \xi_\beta v_{mn}, \tag{2.88}$$

and two for superconformal symmetry,

$$\psi_\alpha^{\text{suconf}} = \sigma^{mn}{}_\alpha{}^\beta \sigma^p{}_{\beta\dot\beta} \overline{\eta}^{\dot\beta} x_p v_{mn}, \tag{2.89}$$

with fermionic collective coordinates ξ and $\overline{\eta}$ [17]. All (multi-)instantons, in any gauge group, have these adjoint fermion zero modes, but the origin of their additional zero modes will not be so easy to interpret.

Chapter 3

Supersymmetry

3.1 Introduction

The field theories of interest in this thesis are not just gauge theories, but specifically *supersymmetric* gauge theories. Supersymmetry is invariance under the exchange of bosonic and fermionic degrees of freedom. Why should we consider it to be of significance?

In particle physics, supersymmetry has been suggested as an explanation for why the Higgs mass is so small compared to the highest known fundamental scale, the Planck mass. Every particle that interacts with the Higgs boson contributes to radiative loop corrections to its mass, which are quadratic in whatever cut-off scale one chooses to represent new physics superceding the standard model (the Planck mass applies, even if there is nothing lower). Using counter-terms to remove such large corrections would require unfeasible fine tuning. Without it, the Higgs would be expected to have a mass of the same order as the Planck mass, and then the mystery is in the relative smallness of the electroweak scale. The same problem does not occur with fermions and gauge particles because their masses are protected by chiral and gauge symmetry respectively. In analogy, supersymmetry protects the masses of scalar particles. The corrections to the Higgs mass from bosons and fermions cancel due to the balance between them in a supersymmetric theory[1]. Furthermore, every currently viable extension to the standard

[1]At the one loop level the cancellation occurs because fermion loops have an extra minus sign compared to loops with bosons. More generally, in a supersymmetric theory, any scalar such as a Higgs boson is paired with a fermion of the same mass. The mass of the fermion is protected by chiral

model incorporates supersymmetry, so it is important to be aware of its consequences.

It is also of interest from a more abstract viewpoint. Knowing that both local gauge symmetry and local Poincaré symmetry (gravity) are of central importance in modern physics, Coleman and Mandula [18] looked for the most general Lie algebra of the symmetries of an S-matrix of a four dimensional relativistic quantum field theory obeying some sensible conditions. Unfortunately, the answer is a direct sum of the Poincaré algebra and the algebra of a compact Lie group. There is no way of intertwining gauge symmetry and gravity in a larger symmetry group from which both naturally occur together, perhaps even with new physics. However, there is a generalisation that does lead to a novel situation. Haag, Łopuszański and Sohnius [19] considered *graded* Lie algebras, which include anticommuting generators, and found the supersymmetry algebra as the most general such object that leaves an S-matrix unchanged. The core of the supersymmetry algebra is illustrated by the anticommutator of the new supersymmetry generators Q_α and $\overline{Q}_{\dot{\alpha}}$, which is proportional to the generator of translations,

$$\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\} = 2\sigma^m{}_{\alpha\dot{\alpha}} P_m. \tag{3.1}$$

If Q_α and $\overline{Q}_{\dot{\alpha}}$ act on a bosonic field they give a fermionic field and vice versa, so it is only when the two types are matched that these operators leave the theory invariant. This is not what has been experimentally observed to date, so why should the theory of fundamental physics not include every possible feature? Perhaps by studying all the alternatives we might find a reason, or discover how supersymmetry can relate to the structure we know of already.

The full supersymmetry algebra is given in the textbook by Wess and Bagger [20], including a discussion of extended supersymmetries where there are many copies of the pair of anticommuting generators introduced above. The extension, or number of supersymmetries, $\mathcal{N}$, is the number of such pairs, and in four dimensions the interesting cases are $\mathcal{N} = 1$, 2 and 4. We will mostly be concerned with the unextended $\mathcal{N} = 1$ case in this thesis. The same reference [20] also contains an excellent introduction to supersymmetric multiplets, superspace, superfields and Lagrangians for supersymmetric field theories. Given such a treatment it is entirely unnecessary to discuss that material at length here, and instead we shall just give a brief summary and orientation in the next

symmetry, and supersymmetry then extends this protection to the scalar mass.

section. After that, we will describe the relevance of supersymmetry to semiclassical calculations, introduce some of the interesting quantities to consider in supersymmetric theories, and look at some of the methods used in attempts to determine them.

3.2 Supersymmetry in field theories

3.2.1 Superspace and superfields

The supersymmetry algebra can be used to show that the operator Q_α increases the spin by one half, therefore at a simplistic level a scalar is connected to a fermion, a fermion to a vector, and so on. The $\mathcal{N} = 1$ supermultiplets of interest in this thesis work in exactly this way.

A chiral or scalar multiplet contains a scalar field $A(x)$ and a fermion $\psi(x)$. Under an infinitesimal supersymmetry transformation, A changes by

$$\delta A = \sqrt{2}\xi\psi, \tag{3.2}$$

where ξ is the Grassmannian parameter of the transformation. Note that the dimensions are correct; equation (3.1) shows that Q_α has mass dimension $\frac{1}{2}$, so we would expect ξ to have the opposite dimension, $-\frac{1}{2}$, in order that they form a dimensionless combination. In four dimensions the mass dimensions of scalars and fermions are 1 and $\frac{3}{2}$ respectively, so the assignment $[\xi] = -\frac{1}{2}$ is in accord with equation (3.2). More generally, this type of argument shows that under supersymmetry, any field gains contributions from fields with mass dimension higher by $\frac{1}{2}$, or n-th derivatives of fields with dimension $n - \frac{1}{2}$ lower. An example of the latter is shown by the infinitesimal transformation of ψ,

$$\delta\psi = \sqrt{2}i\sigma^m\bar{\xi}\partial_m A + \sqrt{2}\xi\mathcal{F}. \tag{3.3}$$

The field $\mathcal{F}$ cannot be a dynamical scalar as it has mass dimension 2, which is too high to allow a kinetic term involving $\mathcal{F}$ in a dimensionless action. It is instead an auxiliary field, which can be eliminated by substituting the solution to its equation of motion[2]. The infinitesimal transformation of $\mathcal{F}$ is

$$\delta\mathcal{F} = \sqrt{2}i\bar{\xi}\bar{\sigma}^m\partial_m\psi, \tag{3.4}$$

[2]Including the auxiliary field $\mathcal{F}$ is a way of linearising the supersymmetry transformations.

which shows that the transformations relating A, ψ and $\mathcal{F}$ are closed. Chiral multiplets are used to describe matter fields in supersymmetric theories; if ψ is a quark then its superpartner A is a scalar quark or squark.

A vector multiplet contains a fermion λ and a gauge field v_m, and also an auxiliary field $\mathcal{D}$. We shall refer to v_m as the gluon field and call λ the gluino. In order for supersymmetry to be valid in any gauge, the gluino must transform in the same representation of the gauge group as the gluon, namely the adjoint representation. The same holds for the matter fields; the squarks must behave in the same way as the quarks under gauge transformations.

A natural language for discussing supersymmetric theories is found in the superspace formalism. The familiar bosonic coordinates x^m are augmented by four Grassmannian coordinates, θ^α and $\overline{\theta}_{\dot{\alpha}}$. Supersymmetry transformations are then essentially translations in this extended space, for example $\theta \mapsto \theta + \xi$, as can be seen from the superspace representation of the supersymmetry generators,

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^m{}_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\partial_m, \tag{3.5}$$

$$\overline{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} + i\theta^\alpha\sigma^m{}_{\alpha\dot{\alpha}}\partial_m. \tag{3.6}$$

These can be used to act on superfields, or fields on superspace. The dependence of these functions on θ and $\overline{\theta}$ can be expressed as a Taylor series that terminates after only a few terms, due to the Grassmannian character of the extra coordinates. The coefficients are conventional fields depending on x, which are bosonic or fermionic according to whether they multiply an even or odd number of Grassmannian coordinates. A supersymmetry transformation has the effect of mixing these component fields amongst each other. However, the most general superfield does not correspond to an irreducible representation of supersymmetry, and to construct more fundamental objects we must impose suitable conditions.

The following derivative operators are covariant with respect to supersymmetry,

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^m{}_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\partial_m, \tag{3.7}$$

$$\overline{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^m{}_{\alpha\dot{\alpha}}\partial_m, \tag{3.8}$$

and can therefore appear in supersymmetrically invariant conditions. For example, we can require that the superfield $\Phi(x,\theta,\overline{\theta})$ vanishes when acted on by one of these operators,

$$\overline{D}_{\dot{\alpha}}\Phi = 0. \tag{3.9}$$

This equation is easily solved by noticing that θ^α and $y^m = x^m + i\theta\sigma^m\overline{\theta}$ are annihilated by $\overline{D}_{\dot{\alpha}}$, so any function of these two variables[3] will fulfil equation (3.9). The most general possibility is

$$\Phi(y,\theta) = A(y) + \sqrt{2}\theta\psi(y) + \theta\theta\mathcal{F}(y). \tag{3.10}$$

The fields A, ψ and $\mathcal{F}$ are exactly the same as those of the chiral multiplet discussed above. Any superfield obeying equation (3.9) is known as a chiral superfield. Note that any product of chiral superfields is also a chiral superfield.

Similarly, $D_\alpha\Phi^\dagger = 0$ implies that $\Phi^\dagger$ is a function of $\overline{\theta}$ and $y^{m\dagger} = x^m - i\theta\sigma^m\overline{\theta}$, and this gives anti-chiral superfields.

Another condition is reality, $V = V^\dagger$, and this leads to a superfield containing the fields λ, v_m and $\mathcal{D}$ that we introduced in the context of the vector supermultiplet earlier, as well as some additional fields. We can generalise gauge transformations, in a manner consistent with the superspace approach, by promoting the gauge transformation parameters to being chiral superfields. These transformations can then be used to set all the fields except the gluon, gluino and auxiliary fields to zero. This choice is called the Wess-Zumino gauge, and it is disturbed by supersymmetry but does not affect the usual gauge redundancy, so conventional gauge transformations can still be performed.

An interesting superfield associated with the vector superfield V is defined by

$$W_\alpha = -\frac{1}{4}\overline{D}\,\overline{D}e^{-V}D_\alpha e^V. \tag{3.11}$$

It is chiral,

$$\overline{D}_{\dot{\alpha}}W_\alpha = 0, \tag{3.12}$$

[3] It is also straightforward to see that Φ does not depend on $\overline{\theta}$ by writing the derivative operators in the $(y, \theta, \overline{\theta})$ coordinate system,

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + 2i\sigma^m{}_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\frac{\partial}{\partial y^m}, \quad \overline{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}}.$$

and obeys the following identity,

$$D^\alpha W_\alpha = \overline{D}_{\dot\alpha}\overline{W}^{\dot\alpha}. \tag{3.13}$$

The component fields of W_α, in the Wess-Zumino gauge or otherwise, are λ, $\mathcal{D}$ and v_{mn}, so it is the supersymmetric generalisation of the gauge field strength. For completeness we list the infinitesimal supersymmetry transformations of these fields,

$$\delta\lambda = i\xi\mathcal{D} + \sigma^{mn}\xi v_{mn}, \tag{3.14}$$

$$\delta v_{mn} = i\xi\left(\sigma_n D_m\overline{\lambda} - \sigma_m D_n\overline{\lambda}\right) + i\overline{\xi}\left(\overline{\sigma}_n D_m\lambda - \overline{\sigma}_m D_n\lambda\right), \tag{3.15}$$

$$\delta\mathcal{D} = \overline{\xi}\overline{\sigma}^m D_m\lambda - \xi\sigma^m D_m\overline{\lambda}. \tag{3.16}$$

The form of equation (3.14) shows that the adjoint fermion zero modes (2.88) are indeed associated with supersymmetry.

3.2.2 Supersymmetric Lagrangians

In order to construct Lagrangians for superfields, we can use the reasoning given in the last section, that any component field incorporates fields of higher mass dimension or derivatives of fields of lower mass dimension, under the influence of supersymmetry. This means that the highest dimension component of any superfield changes by the derivative of some set of fields, as illustrated by equations (3.4) and (3.16)[4]. The integral over space of the highest dimension component must therefore be a supersymmetric invariant, and is a candidate for a term in a supersymmetric Lagrangian. These contributions are classified as F-terms or D-terms, if they are the highest dimension components of chiral or vector superfields respectively.

We can apply the rules of Grassmannian integration, equation (2.36), and define Grassmannian measures on superspace with the following properties[5],

$$\int d^2\theta\, 1 = \int d^2\theta\, \theta^\alpha = 0, \quad \int d^2\theta\, \theta\theta = 1, \tag{3.17}$$

$$\int d^2\overline{\theta}\, 1 = \int d^2\overline{\theta}\, \overline{\theta}_{\dot\alpha} = 0, \quad \int d^2\overline{\theta}\, \overline{\theta}\overline{\theta} = 1. \tag{3.18}$$

[4]Equation (3.16) involves a covariant derivative, but a gauge invariant object suitable for inclusion in a Lagrangian would change by a simple coordinate derivative.

[5]The product $\theta\theta = \theta^\alpha\theta_\alpha$ is equal to $-2\theta^1\theta^2$, so the conditions (3.17) imply $d^2\theta = -\frac{1}{2}d\theta^2 d\theta^1$.

Then the F-terms and D-terms can be extracted from their superfields by integration. For example, the kinetic terms in the action for a chiral superfield come from a D-term,

$$\int d^4x \int d^2\theta \, d^2\overline{\theta} \, \Phi^\dagger \Phi. \tag{3.19}$$

Chiral superfields do not depend on $\overline{\theta}$, so to get an F-term, such as the one that gives the kinetic terms for a vector superfield, we only have to integrate over half of the Grassmannian coordinates,

$$\int d^4x \, \frac{1}{g^2} \mathrm{Re} \left(\int d^2\theta \, \mathrm{Tr} \, W^\alpha W_\alpha \right). \tag{3.20}$$

Note that $W^\alpha W_\alpha$ is a chiral superfield because it is a product of chiral superfields.

3.2.3 Supersymmetric Hamiltonians

The supersymmetry algebra, equation (3.1), allows us to find a general, formal expression for the Hamiltonian in a supersymmetric field theory,

$$\begin{aligned} H = P^0 &= \frac{1}{2} \overline{\sigma}^{0\dot{\alpha}\alpha} \sigma^m{}_{\alpha\dot{\alpha}} P_m = \frac{1}{4} \overline{\sigma}^{0\dot{\alpha}\alpha} \left\{ Q_\alpha, \overline{Q}_{\dot{\alpha}} \right\} \\ &= \frac{1}{4} \left(Q_1 \overline{Q}_{\dot{1}} + Q_2 \overline{Q}_{\dot{2}} + \overline{Q}_{\dot{1}} Q_1 + \overline{Q}_{\dot{2}} Q_2 \right). \end{aligned} \tag{3.21}$$

The expectation value of the energy in any state $|\Omega\rangle$ is therefore positive semi-definite,

$$\langle \Omega | H | \Omega \rangle = \frac{1}{4} \left(\left| \overline{Q}_{\dot{1}} |\Omega\rangle \right|^2 + \left| \overline{Q}_{\dot{2}} |\Omega\rangle \right|^2 + \left| Q_1 |\Omega\rangle \right|^2 + \left| Q_2 |\Omega\rangle \right|^2 \right) \geqslant 0, \tag{3.22}$$

and zero only in a supersymmetric state,

$$\langle \Omega | H | \Omega \rangle = 0 \quad \Leftrightarrow \quad \overline{Q}_{\dot{\alpha}} |\Omega\rangle = Q_\alpha |\Omega\rangle = 0. \tag{3.23}$$

In any physical Hamiltonian there will be both kinetic and potential terms. The kinetic energy being equal to zero is obviously a necessary condition for the total energy to vanish. The cases where the contributions to the potential due to F-terms or D-terms become zero are called F-flatness and D-flatness respectively.

3.3 Supersymmetry in semiclassical calculations

There are some results and theorems due to supersymmetry which are particularly helpful in semiclassical calculations. To see how and why, we must first consider some of the problems that can arise in a generic semiclassical calculation.

After removing the divergences due to the existence of bosonic zero modes (by including massless fermions in the theory), and regulating and renormalising the ultraviolet divergences of the determinants, the main obstacle to performing semiclassical calculations is that in most situations there is no control available on the convergence of the perturbative expansion.

In perturbative QCD, for comparison, the expansion parameter is the renormalised coupling $g^2(\mu)$, where μ is set by an appropriate physical scale. The only way of ensuring that $g^2(\mu) < 1$ is to restrict attention to scattering processes at high energies, where the negative β function drives the coupling to a small value. Quantities of interest in the low energy regime where $g^2(\mu)$ is of order unity or greater, such as the properties of bound states, are incalculable by perturbative means.

With instantons, the natural scale is the reciprocal of the instanton size, ρ^{-1}. However, it is necessary to integrate over all values of ρ, and therefore to include the infrared effects from large instantons, associated with a large coupling. The problems this brings show up in the one instanton measure [7, 21], the ρ component of which is proportional to

$$\int_0^\infty d\rho\, \rho^p, \tag{3.24}$$

with p a positive constant. This is divergent at large sizes, unless there is a factor in the integrand which renders it finite.

Another problem with instanton calculations is that even if one can find a regime where the coupling is small so the procedure is well defined, the instanton effects are still linked to the tiny factor $\exp\left(-\frac{8\pi^2}{g^2}\right)$, and are therefore insignificant compared to conventional perturbative effects in their common region of validity.

Supersymmetry provides the solution to both of these problems, and also leads to simplifications and improves the semiclassical approximation so that *exact* results may be found. The powerful tool used to achieve this is a non-renormalisation theorem [22], which states that the perturbative corrections to any F-term must vanish. To see why, let us write the result of a generic field theory calculation at small coupling as

$$a_0 + a_1 g^2 + a_2 g^4 + \dots , \tag{3.25}$$

where if, for example, this is the (multi-)instanton contribution to a correlation function, all of the coefficients $\{a_n\}$ would be proportional to $\exp\left(-\frac{8\pi^2|k|}{g^2}\right)$. Every coefficient with $n \geqslant 1$ derives from loop corrections that, in a supersymmetric theory, naturally contain an integration over a supersymmetric version of momentum space, $\int d^4k\, d^2\theta\, d^2\overline{\theta}$. The integration must be over the whole of this superspace, not just $\int d^4k\, d^2\theta$ alone, because perturbation theory respects chiral symmetry (rotations of the phase of fermions). However, chiral superfields do not depend on $\overline{\theta}$, and so the integration over $\int d^2\overline{\theta}$ will give zero. Therefore every a_n identically vanishes for $n \geqslant 1$.

This theorem does not extend to non-supersymmetric theories; without superspace there is no concept of an F-term. However, in the supersymmetric case it means that restricting the instanton calculation to one loop level still gives the full answer for certain quantities.

A further consequence of the non-renormalisation theorem is that the conventional perturbative corrections to such a quantity vanish, so in the case that the F-term is classically zero, its quantum value is determined entirely by instantons with no competition from conventional perturbation theory!

Even though the perturbative series is trivial for F-terms, it is still not possible to calculate them unless the coupling constant is small. However, we expect all correlation functions in a supersymmetric theory to be holomorphic in the coupling constant, so if there is a method of calculating them with small coupling ensured, then their large coupling forms can be recovered by analytic continuation. Extended supersymmetries allow control over the size of the coupling; the higher the extension the greater the control. Exact calculations of the type discussed above have been performed in theories with both possible versions of extended supersymmetry.

In $\mathcal{N} = 4$ supersymmetric theories, the β function vanishes so the theory is in fact superconformal. The coupling is then just a constant which can be set to any value. Instanton calculations [23, 24] in this type of theory have provided evidence for the AdS/CFT duality conjectured by Maldacena [25].

In $\mathcal{N} = 2$ theories, the β function is negative, but the field content includes a scalar, which can gain a VEV that spontaneously breaks the gauge group to its maximal abelian subgroup. In this Coulomb phase, the theory becomes weakly coupled if the VEV is chosen sufficiently large. Another way of seeing that this is of help is that the scalar field contributes a quantity proportional to ρ^2 to the action, and when exponentiated this leads to a damping factor that improves the convergence of the instanton measure. The low energy dynamics of $\mathcal{N} = 2$ theories were analysed by Seiberg and Witten [26, 27], using the assumption of symmetry under electric-magnetic duality in the low energy theory. Their exact results were checked by explicit instanton calculations in [28, 29] and references therein.

For the case of $\mathcal{N} = 1$ supersymmetry, under consideration in this thesis, there is no automatic means to keep the coupling constant small, but we shall discuss various applicable methods in the remaining sections of this chapter, and in chapter 4.

In addition to the features above that make semiclassical calculations possible, supersymmetry provides theorems that can vastly simplify the computations. The most important is that in a theory in four dimensions, the determinants of non-zero eigenvalues exactly cancel between fermionic and bosonic degrees of freedom[6] [30], so only the zero modes have to be considered. Another theorem that will be relevant in this thesis is that any correlation function involving only the lowest components of chiral superfields is a constant, and so does not depend on the positions of the fields in the multi-point function [31].

3.4 Supersymmetric Yang-Mills theory and the gluino condensate

To begin the study of supersymmetric gauge theories, we can start with the simplest possible example. This is a theory without matter fields (a pure gauge theory), and with the smallest interesting gauge group. This must be a compact Lie group, but there are are no instantons in a $U(1)$ theory, so we also require a non-abelian gauge group, of which the lowest dimension example is $SU(2)$.

The field content of $\mathcal{N} = 1$ supersymmetric pure $SU(2)$ Yang-Mills theory is the three gauge or gluon fields, v_m^a, and their superpartners the gluinos, λ_α^a. These together

[6] This is analogous to the cancellation of scalar mass corrections in a supersymmetric theory.

form a vector superfield, and the action for this superfield is

$$S = \int d^4x \, \frac{1}{2g^2} \mathrm{Tr} \, \left(\int d^2\theta \, W^\alpha W_\alpha + \int d^2\overline{\theta} \, \overline{W}_{\dot{\alpha}} \overline{W}^{\dot{\alpha}} \right). \tag{3.26}$$

Written out using the component fields, this is

$$S = \int d^4x \, \frac{1}{g^2} \mathrm{Tr} \, \left(\frac{1}{2} v_{mn} v^{mn} + 2i\overline{\lambda}\overline{\sigma}^m D_m \lambda + \mathcal{D}^2 \right). \tag{3.27}$$

Recall that λ necessarily transforms in the adjoint representation of the gauge group, so the covariant derivative acts on it as $D_m \lambda = \partial_m \lambda - i[v_m, \lambda]$. We can see from the action that this theory may be viewed in non-supersymmetric terms as a gauge theory with one adjoint fermion. A theta term can be included by modifying the action to

$$S = \int d^4x \, \mathrm{Im} \left(\frac{\tau}{4\pi} \int d^2\theta \, \mathrm{Tr} \, W^\alpha W_\alpha \right), \tag{3.28}$$

where τ, defined by

$$\tau = \frac{4\pi i}{g^2} + \frac{\vartheta}{2\pi}, \tag{3.29}$$

is known as the complexified coupling constant. This action contains the previous terms plus

$$S_\vartheta = -\frac{i\vartheta}{16\pi^2} \int d^4x \, \mathrm{Tr} \, (v_{mn} {*v}^{mn}). \tag{3.30}$$

The formulae above are in fact applicable to supersymmetric pure Yang-Mills theory with any simple gauge group.

The first correlation function to consider in these theories is the gluino condensate[7],

$$\left\langle \frac{\mathrm{Tr} \, \lambda\lambda}{16\pi^2} \right\rangle. \tag{3.31}$$

This is of interest for the following reasons.

- It is Lorentz and gauge invariant, so its value is not arbitrary.
- It is proportional to the lowest component of the chiral superfield $\langle \mathrm{Tr} \, W^\alpha W_\alpha \rangle$, and therefore, by the theorem stated in the last section, will be a constant, which would be expected to be easier to calculate than a quantity with spatial dependence. Actually, since the condensate only depends on one point in space, the fact that it

[7] The factor $16\pi^2$ is a conventional normalisation.

is constant also follows from translational invariance. By dimensional analysis, it must have the form $c\Lambda^3$, where Λ is the dynamically generated scale in whatever regularisation scheme is being used, so all that remains to be determined is the dimensionless constant c.

- Gauge theories with $\mathcal{N} = 1$ supersymmetry are the closest amongst supersymmetric theories to gauge theories of relevance to present experiments, for example QCD, so they have the most similar behaviour. We can attempt to gain understanding of aspects of non-supersymmetric theories, such as confinement, by working with the supersymmetric theories and using supersymmetry to allow analytic computations that are not possible in the non-supersymmetric cases. In particular, the gluino condensate is not an invariant under chiral symmetry, so a non-zero value is a signal of spontaneous chiral symmetry breaking, a phenomenon also thought to occur in QCD at low energies. We will show later how the gluino condensate is also linked to confinement in $\mathcal{N} = 1$ supersymmetric theories (see chapter 5).

  The condensate of the superpotential W_α additionally contains $\langle \mathrm{Tr}\, v_{mn} v^{mn} \rangle$, multiplied by $\theta\theta$. This gluon condensate might be thought to have greater relevance, since it also appears in non-supersymmetric gauge theories. However, invariance under translations in superspace shows that it must be zero in a theory with unbroken supersymmetry, so we cannot gain any more insight into non-supersymmetric gluodynamics this way.

- The gluino condensate is part of a correlation function of a chiral superfield, so the non-renormalisation theorem discussed in section 3.3 can be applied to it, and it may be possible to calculate it exactly. Furthermore, the gluino condensate is zero to all orders in conventional perturbation theory, which respects chiral symmetry, so its value is determined entirely by semiclassical contributions.

As a quantity of obvious interest, for all the reasons above, it was calculated early on in the study of supersymmetric gauge theories [32, 33, 34, 31, 35, 36, 37], but without consistent results. The controversy over the value of the gluino condensate [35, 33, 31, 38, 39] has lasted nearly 15 years! The work we will present in this thesis should finally resolve the dispute, but first we must examine the original methods used to calculate the gluino condensate.

3.4.1 The SCI approach

The very first method used to calculate the gluino condensate was as direct as possible. Let us consider first the $SU(2)$ theory, in which case the one instanton has four adjoint fermion zero modes, the supersymmetric and superconformal modes introduced in section 2.2.5. Clearly, in a straightforward instanton calculation of the gluino condensate, the fermion zero modes will not be saturated and the result will be zero. However, in $SU(2)$ gauge theory one can consider the correlation function

$$\left\langle \frac{\operatorname{Tr}\lambda\lambda(x)}{16\pi^2}\frac{\operatorname{Tr}\lambda\lambda(0)}{16\pi^2}\right\rangle. \tag{3.32}$$

This is also a candidate for the use of the non-renormalisation theorem, and it may have a non-zero value in a one instanton calculation, because it contains four fermionic insertions. It is again a correlation function made of the lowest components of chiral superfields, so it must be a constant, not even dependent on $|x|$. We can then invoke cluster decomposition, which is an axiom of field theory that states that in the limit of large $|x|$, any correlation function of the form $\langle F(x)G(0)\rangle$ should tend to $\langle F\rangle\langle G\rangle$ plus exponentially small corrections, to find the gluino condensate as the square root of the one instanton answer. Using the Pauli-Villars regularisation scheme, this procedure yields [32, 33]

$$\left\langle \frac{\operatorname{Tr}\lambda\lambda}{16\pi^2}\right\rangle = \pm\sqrt{\frac{4}{5}}\Lambda^3. \tag{3.33}$$

This method can be generalised to any gauge group. In order to saturate the $2c_2$ adjoint fermion zero modes, one must calculate

$$\left\langle \prod_{n=1}^{c_2} \frac{\operatorname{Tr}\lambda\lambda(x_n)}{16\pi^2}\right\rangle. \tag{3.34}$$

and then take the c_2-th root. For gauge group $SU(N)$, which has $c_2 = N$, the result is [34, 31]

$$\left\langle \frac{\operatorname{Tr}\lambda\lambda}{16\pi^2}\right\rangle = \left(\frac{2^N}{(N-1)!(3N-1)}\right)^{\frac{1}{N}}\Lambda^3. \tag{3.35}$$

The value of the gluino condensate always contains a c_2-th root of unity, which shows that there must be c_2 vacua with distinct but physically equivalent versions of the gluino condensate. This agrees with predictions made by Witten [40], and will be seen in more detail in chapters 5 and 6.

This approach to calculating the gluino condensate is called SCI, or strong coupling instanton, as there is no attempt made to control the effects of the large coupling constant from the unbroken non-abelian gauge group. Whilst the non-renormalisation theorem might seem to suggest that a small coupling is not necessary, since there is no perturbative expansion to be convergent or divergent, there is possible cause for concern. We are only guaranteed that instantons are the sole relevant semiclassical configurations in the weak coupling regime. Cluster decomposition applies to full correlation functions, not partial contributions to them, so if we are neglecting some unknown configurations that are important at large coupling, the value of the gluino condensate derived by the SCI method will be in error. This is in fact exactly the case, as we shall discuss at the start of chapter 4.

3.4.2 The WCI approach

In the SCI approach, the gluino condensate is calculated from a somewhat circuitous route, via a more complicated correlation function and cluster decomposition. There are several other methods available, which also do not lead to the gluino condensate directly, but have the advantage of ensuring that the coupling constant remains small. This is achieved by considering a family of theories, labelled by some physical parameter that influences the coupling constant. One value of the parameter will give the $\mathcal{N} = 1$ supersymmetric gauge theory under investigation, all the others will represent modified versions of this theory, including, in one limit, some where the coupling is forced to be small. This gives rise to the name WCI, or weak coupling instanton, for this type of approach. We can reliably calculate the gluino condensate in the small coupling constant limit, and then rely on supersymmetry, which implies that the result is holomorphic in the coupling, and therefore also in the related parameter. This enables us to return to the case of interest.

The original example of a WCI calculation of the gluino condensate was to modify the $SU(2)$ theory by adding matter in the fundamental representation of the gauge group [35]. To maintain supersymmetry we must add a chiral multiplet with a scalar as well as a fermionic field. The scalar ϕ can have a VEV, which has two effects. The first is to spontaneously break the gauge group completely and, for large values of the VEV, keep the coupling small. Therefore, the VEV is the parameter which continuously modifies the theory in this case. The second consequence of a non-zero VEV is that most of the symmetries leading to fermionic zero modes are destroyed. For instance, the superconformal zero modes for the gluino will not exist because the VEV breaks the scale invariance component of conformal symmetry. For a one instanton, only the two supersymmetric modes remain, since they are protected by the unbroken supersymmetry. This configuration can contribute directly to the gluino condensate in this situation, and is furthermore the only configuration that may do so.

However, we still do not calculate the gluino condensate directly, as we wish to be able to relate it easily to the gluino condensate in the pure theory. Instead we consider

$$\left\langle \frac{\mathrm{Tr}\,\lambda\lambda(x)}{16\pi^2}\,\phi^2(0)\right\rangle . \tag{3.36}$$

This correlation function is, once more, made from the lowest components of chiral superfields, and so is equal to a constant, namely

$$\left\langle \frac{\mathrm{Tr}\,\lambda\lambda(x)}{16\pi^2}\,\phi^2(0)\right\rangle = \Lambda_1^5, \tag{3.37}$$

where Λ_1 is the dynamically generated scale in the theory with one flavour of matter in the fundamental representation. We can again use cluster decomposition to infer that

$$\left\langle \frac{\mathrm{Tr}\,\lambda\lambda}{16\pi^2}\right\rangle \langle\phi^2\rangle = \Lambda_1^5. \tag{3.38}$$

In order to extract the gluino condensate in the $SU(2)$ theory without matter, we must give the matter superfield a large mass m and decouple it from the theory using the renormalisation group. Correlation functions in supersymmetric theories should be holomorphic in the masses of the fields, which implies that the above result is unchanged by making ϕ massive. With the use of the Konishi anomaly relation [41],

$$\left\langle \frac{\mathrm{Tr}\,\lambda\lambda}{16\pi^2}\right\rangle = m\,\langle\phi^2\rangle\,, \tag{3.39}$$

and the renormalisation group decoupling equation,

$$\Lambda^6 = m\Lambda_1^5, \tag{3.40}$$

we can find the WCI value for the gluino condensate in $\mathcal{N} = 1$ supersymmetric pure $SU(2)$ gauge theory,

$$\left\langle \frac{\operatorname{Tr}\lambda\lambda}{16\pi^2} \right\rangle = \pm\Lambda^3. \tag{3.41}$$

Other examples of WCI calculations involve deriving the gluino condensate from a limit of the ADS superpotential, which is discussed in section 6.2, or using Seiberg and Witten's solution for the low energy effective action of $\mathcal{N} = 2$ supersymmetric gauge theories, as reviewed in [42]. Mass terms are added to break the extended supersymmetry, so that the mass acts as the controlling parameter in this case. The same result is found from all WCI calculations, which is a good consistency check, as all WCI approaches have similar strategies but are completely independent in the details of the calculations.

WCI methods have also given the gluino condensate for all classical gauge groups[8] [36, 37, 43],

$$\left\langle \frac{\operatorname{Tr}\lambda\lambda}{16\pi^2} \right\rangle_{SU(N)} = \Lambda^3, \tag{3.42}$$

$$\left\langle \frac{\operatorname{Tr}\lambda\lambda}{16\pi^2} \right\rangle_{SO(N)} = \frac{1}{2}\cdot 2^{\frac{4}{N-2}}\Lambda^3, \tag{3.43}$$

$$\left\langle \frac{\operatorname{Tr}\lambda\lambda}{16\pi^2} \right\rangle_{USp(N)} = 2\cdot 2^{-\frac{2}{N+1}}\Lambda^3. \tag{3.44}$$

Note that the SCI and WCI predictions for the gluino condensate in an $SU(N)$ theory disagree more as N increases.

An imaginative proposal was made by Kovner and Shifman [38], in an attempt to resolve this discrepancy. They postulated the existence of an extra vacuum state, with vanishing gluino condensate, motivated by the intuitive reasoning that an instanton does not contribute directly to the gluino condensate because it averages over all the vacua, and the sum over roots of unity vanishes. If the chirally symmetric Kovner-Shifman vacuum exists, then for $SU(2)$, the two point function in equation (3.36)

[8]See also table 6.2 and appendix B. The c_2-th roots of unity are considered implicit in these formulae.

would be averaged over the two conventional vacua where it should have value Λ^6, and the extra vacuum where it is zero. If the probabilities of finding the theory in each vacuum are arranged correctly, the result $\frac{4}{5}\Lambda^6$ can be obtained. Unfortunately, this is not a very efficient mechanism, since it requires the existence of a new vacuum, plus a highly specific organisation of the probabilities assigned to each vacuum, which is assumed rather than derived, and also elaborate explanations for why the Kovner-Shifman vacuum does not contribute to Witten's index. More definitely, however, it has been shown to be unviable, in [42].

In this thesis we shall describe a more robust explanation for the difference between the SCI and WCI results. It involves another type of WCI calculation, where the modification is to consider the theory on the cylinder $\mathbb{R}^3 \times S^1$, instead of on $\mathbb{R}^4$. The new parameter is the radius of the circle; the coupling constant is small when the radius is small, and as the radius tends to infinity the theory becomes indistinguishable from the version without a compactified coordinate. This alteration requires us to consider configurations other than Yang-Mills instantons, but brings the advantages of allowing direct evaluation of the gluino condensate and providing a way of viewing the configurations neglected in the SCI approach. We shall consider first the simplest theory, with gauge group $SU(2)$, in chapter 5. The generalisation to any other gauge group is relatively straightforward in this formalism, so in chapter 6 we confirm the classical gauge group results above and also predict the values of the gluino condensate in all exceptional groups.

Chapter 4

The cylinder and monopoles

4.1 Introduction

In this chapter we shall describe the idea behind the work presented here with the aim of resolving the discrepancy between the SCI and WCI values of the gluino condensate. The clue we already have is suspicion of the SCI calculation because there is the possibility of neglected configurations in the strong coupling regime, as explained in section 3.4.1. This doubt is justified, because explicit calculations [42] have shown that instanton contributions violate cluster decomposition, which implies that they alone do not give the full correlation functions. However, in order to confirm this hypothesis, we must identify the missing configurations, understand how they relate to instantons, and show that they generate the required contributions to correlation functions.

We can take inspiration for what the extra configurations may be from the early literature containing investigations of semiclassical effects in field theories [44, 45]. Instantons were conjectured to behave as composite objects when the coupling constant is large, and break up into instanton partons[1]. This was of interest because, in three dimensional theories, the relevant instanton solutions had been shown to cause confinement [47], and it was hoped to extend this analysis and prove that confinement also occurs in four dimensions. However, in the three dimensional case the instantons are the same configurations that are monopoles in a four dimensional theory (finite action in three dimensions corresponds to finite energy or mass in four; more details about

[1] Ionization of this kind had been seen to occur in a two dimensional theory [46].

this connection may be found in [4, 48] and section 4.3). The strong Coulomb interactions between monopoles and anti-monopoles are an integral part of the confinement mechanism in [47]; see the Discussion in [1]. Four dimensional instantons and anti-instantons in $SU(2)$ can be shown to have dipole interactions[2] [44], which are too weak to generate a similar effect. However, if instantons were to dissociate into component poles, then those might be suitable objects to bring about confinement. In an attempt to find candidates for instanton partons, solutions to the Yang-Mills equations with appropriate properties were constructed [44]. They are called merons, and have fractional topological charge, and correspondingly infinite action. Unfortunately, they are also non-continuous, and so altogether are very challenging configurations to manipulate or apply in meaningful calculations. We employ a different strategy here; by modifying space from $\mathbb{R}^4$ to the four-dimensional cylinder $\mathbb{R}^3 \times S^1$ (imposing periodicity along one direction), we can identify the instanton partons with well-behaved, well-known solutions, and gain control over both the infinite action of the partons and the size of the coupling constant. This therefore leads to a WCI method that demonstrates the shortcomings of the SCI approach. Furthermore, this scheme is of particular interest to us because of the fractional topological charge of the partons, which means they have two adjoint fermion zero modes apiece and so can contribute directly to the gluino condensate.

We devote most of this chapter to an investigation of the finite action classical solutions on $\mathbb{R}^3 \times S^1$. They were classified by Gross, Pisarski and Yaffe [49] in the context of non-zero temperature field theory, which is formulated on the same space. However, it also involves anti-periodic boundary conditions for fermions, compared to periodic ones for bosons, which would break supersymmetry. Instead, we are interested in the theory on the cylinder, which has periodic boundary conditions for both bosons and fermions,

$$v_m(x^0, x^\mu) = v_m(x^0 + 2\pi R, x^\mu), \quad \lambda_\alpha(x^0, x^\mu) = \lambda_\alpha(x^0 + 2\pi R, x^\mu), \tag{4.1}$$

where R is the radius of the circle. Nevertheless, the classification refers to the bosonic fields and so applies to this case as well. It divides solutions into monopoles and calorons, which we shall describe in turn, first for gauge group $SU(2)$ and then for any

[2] In general, these are c_2-pole interactions, where c_2 is the dual Coxeter number of the gauge group.

gauge group. Note that although these are all instantons on $\mathbb{R}^3 \times S^1$, in the sense of being finite minima of the action, we shall reserve the term for Yang-Mills instantons on $\mathbb{R}^4$. Monopoles, calorons and instantons may all be referred to as semiclassical configurations.

4.2 Monopoles in Yang-Mills-Higgs theory

We shall begin by discussing monopole solutions in the context in which they were first found; Yang-Mills-Higgs theory, an $SU(2)$ gauge theory in Minkowski spacetime with a scalar matter field ϕ transforming in the adjoint representation. The original motivation for choosing this particular field content was that the scalar acts as a Higgs field, gaining a VEV, spontaneously breaking the gauge group to $U(1)$, and enabling the study of a QED-like theory where the abelian gauge group is embedded in a compact group. This can be seen from the action of the theory,

$$S = \int d^4x \left(-\frac{1}{2g^2} \mathrm{Tr}\,(v_{mn}v^{mn}) - \frac{1}{g^2} \mathrm{Tr}\,(D_m\phi D^m\phi) - V(\phi) \right), \tag{4.2}$$

where $V(\phi)$ may be considered to be the usual quartic potential,

$$V(\phi) = \frac{\lambda}{4} \left(|\phi|^2 - u^2\right)^2, \tag{4.3}$$

with $|\phi|^2 = 2\mathrm{Tr}\,\phi^2$. However, the form of the potential is irrelevant except that it must be gauge invariant, positive semi-definite, and have a single minimum of zero, at which point we say

$$|\phi|^2 = u^2. \tag{4.4}$$

Any value of ϕ that solves this is a possible VEV, $\langle\phi\rangle$, but whereas u is gauge invariant, $\langle\phi\rangle$ is not, and may for example be rotated by a gauge transformation into the form

$$\langle\phi\rangle = -u\frac{\tau_3}{2}. \tag{4.5}$$

In this case the unbroken $U(1)$ subgroup is $\{e^{i\Omega\frac{\tau_3}{2}}\}$.

Monopoles appear as solitons in this theory, that is stable configurations associated with an energy density that is localised in space, which therefore have some of the properties of particles. In this case they also carry magnetic charge, despite the fact that the particles associated with quantum excitations of the same gauge fields are only electrically charged.

We therefore seek time independent solutions, and so instead of attempting to solve the equations of motion directly, we can minimise the static energy, or mass, of the monopole. With the gauge choice $v_0 = 0$, this can be written as

$$\mathcal{E} = \int d^3x \left(\frac{1}{g^2} \mathrm{Tr}\left[(D_\mu \phi)^2 + (B_\mu)^2 \right] + V(\phi) \right) \tag{4.6}$$

where $B_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho} v^{\nu\rho}$. We are only interested in minima where the value of the energy is finite, which implies that the potential V must tend to zero at spatial infinity, so the scalar field of the monopole solution must tend to the VEV in that limit. If we write $\phi = \phi^a \frac{\tau_a}{2}$, the condition for the potential to vanish, equation (4.4), becomes

$$(\phi^1)^2 + (\phi^2)^2 + (\phi^3)^2 = u^2, \tag{4.7}$$

which is the equation of a sphere, S^2. Therefore the scalar field of a finite mass monopole provides a mapping from the sphere at infinity of $\mathbb{R}^3$ to the sphere of Higgs vacua. Analogously to the topological classification derived from the boundary conditions of finite action Yang-Mills instantons, the equivalence classes of these mappings are in one-to-one correspondence with the elements of a homotopy group. In this case it is

$$\pi_2(S^2) = \mathbb{Z}, \tag{4.8}$$

so every monopole solution may also be labelled with an integer.

The quantity $\frac{2}{g|\phi|}\mathrm{Tr}\,(\phi B^\mu)$ is the magnetic field of the unbroken $U(1)$, so the magnetic charge of a solution is given by the integral

$$q = \int_{S^2_\infty} (d^2x)_\mu \frac{2}{gu} \mathrm{Tr}\,(\phi B^\mu) \tag{4.9}$$

Also, a more detailed analysis of the asymptotic form of any monopole solution shows that this integral gives the winding number k from the homotopy group above, multiplied by $\frac{4\pi}{g}$. Therefore, the possible magnetic charges of the solutions are quantized, in accordance with Dirac quantisation [50].

We can minimise the energy within each topological sector, leading to what are known as 't Hooft-Polyakov monopoles [51, 52]. There is no analytic expression for the monopole fields, their form is only known numerically, but we may find a bound for their masses. Using a Bogomol'nyi argument similar to that in section 2.2.2, but applied to the energy in this case, we find

$$\mathcal{E} = \int d^3x \left(\frac{1}{g^2} \mathrm{Tr}\,[D_\mu \phi \mp B_\mu]^2 \pm \frac{2}{g^2} \mathrm{Tr}\,(D_\mu \phi) B^\mu + V(\phi) \right) \tag{4.10}$$

$$\geqslant \pm \int d^3x \, \frac{2}{g^2} \mathrm{Tr}\,(D_\mu \phi) B^\mu. \tag{4.11}$$

(Equation (4.6) shows that $\mathcal{E} \geqslant 0$, so the sign above should be chosen to give a non-trivial bound.) We can evaluate this further through integration by parts, which leads to

$$\mathcal{E} \geqslant \pm \int_{S^2_\infty} (d^2x)_\mu \, \frac{2}{g^2} \mathrm{Tr}\,(\phi B^\mu) \tag{4.12}$$

The mass of a monopole is therefore bounded by a combination of the magnetic charge and the VEV,

$$\mathcal{E} \geqslant \pm \frac{qu}{g} = \frac{4\pi u |k|}{g^2}. \tag{4.13}$$

The bound is attained if the Bogomol'nyi equations,

$$B_\mu = \pm D_\mu \phi, \tag{4.14}$$

hold, and the potential is set to zero. The latter condition is called the BPS limit after Bogomol'nyi [53], and Prasad and Sommerfield [54] who first considered it. It corresponds to setting λ to zero in the example of the quartic potential above, and leaving the vacuum condition $|\phi|^2 = u^2$ as the only remaining effect of the potential. In this situation, we can find an analytic expression for the monopole fields with $k = 1$, partly because the first order equations (4.14) are much easier to solve than the full second order equations of motion. This solution is called the BPS one monopole, and we present it in the next section.

4.2.1 The BPS one monopole

A monopole solution in the BPS limit with winding number unity, and magnetic charge $\frac{4\pi}{g}$, is given by

$$\phi^a = -\frac{x^a}{|x|^2} \left(u|x| \coth(u|x|) - 1 \right), \tag{4.15}$$

$$v^a_\mu = \epsilon_{a\mu\nu} \frac{x^\nu}{|x|^2} \left(1 - \frac{u|x|}{\sinh(u|x|)} \right), \tag{4.16}$$

where $|x| = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$ and $\phi = \phi^a \frac{\tau_a}{2}, v_\mu = v_\mu^a \frac{\tau_a}{2}$. In order to show that this is a solution, let us define

$$\gamma = u|x|, \quad a(\gamma) = \frac{\gamma}{\sinh\gamma}, \quad b(\gamma) = \gamma \coth\gamma, \tag{4.17}$$

so that

$$\phi^a = -\frac{x^a}{|x|^2}(b-1), \quad v_\mu^a = \epsilon_{a\mu\nu}\frac{x^\nu}{|x|^2}(1-a). \tag{4.18}$$

For any function of γ, $f(\gamma)$, we have

$$\partial_\mu f = \frac{x_\mu}{|x|^2}\left(\gamma\frac{df}{d\gamma}\right), \tag{4.19}$$

and in particular, for the functions a and b,

$$\gamma\frac{db}{d\gamma} = b - a^2, \quad \gamma\frac{da}{d\gamma} = a - ab. \tag{4.20}$$

Now we can find that

$$(D_\mu\phi)^a = \partial_\mu\phi^a + \epsilon^{abc}v_{\mu b}\phi_c \tag{4.21}$$

$$= \frac{\delta_\mu^a}{|x|^2}(a - ab) + \frac{x_\mu x^a}{|x|^4}(a^2 + ab - a - 1), \tag{4.22}$$

and

$$B_\mu^a = \epsilon_{\mu\nu\rho}\left(\partial^\nu v^{\rho a} + \frac{1}{2}\epsilon^{abc}v_b^\nu v_c^\rho\right) \tag{4.23}$$

$$= \frac{\delta_\mu^a}{|x|^2}(a - ab) + \frac{x_\mu x^a}{|x|^4}(a^2 + ab - a - 1), \tag{4.24}$$

so indeed $B_\mu = D_\mu\phi$. Also, using the asymptotic behaviour $a \to 0$, $b \to \gamma$ as $|x| \to \infty$, it is easy to check that

$$\int_{S_\infty^2} (d^2x)_\mu \frac{2}{gu}\mathrm{Tr}\,(\phi B^\mu) = \frac{4\pi}{g} \tag{4.25}$$

for this monopole.

The expressions given above are not the fields of the most general BPS one monopole. However, all the other $k = 1$ solutions can be found by translating the monopole (replacing x by $x - X$), or performing global transformations from the unbroken $U(1)$, which will not disturb the VEV. These latter modifications can be displayed most easily by acting on the solution above with a singular gauge transformation so that

$\langle\phi\rangle = -u\frac{x^a}{|x|}\frac{\tau_a}{2}$ is mapped to $\langle\phi\rangle = -u\frac{\tau_3}{2}$, then the relevant global transformations are $\{e^{i\Omega\frac{\tau_3}{2}}\}$. The original solution, with a VEV that follows the direction of the radial vector, is said to be written in a hedgehog or regular gauge, when the VEV is aligned in a single direction the monopole is in a singular gauge.

4.2.2 Multi-monopoles

The BPS solution is a one monopole, but solutions to the Bogomol'nyi equations with all values of the winding number exist, and are known in general as multi-monopoles. The sign in equation (4.14) must match the sign of the winding number; solutions with negative magnetic charge are called anti-monopoles and are easily obtained from solutions with positive magnetic charge by sending $\phi \mapsto -\phi$. A monopole with winding number k depends in general on $4|k|$ parameters [55].

4.3 Monopoles as semiclassical configurations

In order to find minimal action configurations on $\mathbb{R}^3 \times S^1$, we might begin our search by considering fields that are independent of the periodic coordinate, x^0. The topological arguments of section 2.2.1 do not apply to the modified space, but the Bogomol'nyi bound from section 2.2.2 is still relevant and shows that we should investigate (anti-)self-dual configurations. The conditions $*v_{mn} = \pm v_{mn}$ can be rewritten as

$$v_{23} = \mp v_{01}, \quad v_{31} = \mp v_{02}, \quad v_{12} = \mp v_{03}, \tag{4.26}$$

and if none of the fields depends on x^0, then these are equivalent to

$$v_{23} = \pm D_1 v_0, \quad v_{31} = \pm D_2 v_0, \quad v_{12} = \pm D_3 v_0. \tag{4.27}$$

We can also rewrite the Bogomol'nyi equations in the form

$$v_{23} = \pm D_1 \phi, \quad v_{31} = \pm D_2 \phi, \quad v_{12} = \pm D_3 \phi, \tag{4.28}$$

so if we identify[3] $v_0 = \phi$, then any (anti-)monopole in the BPS limit is an (anti-)self-dual gauge configuration. Their action is equal to $\int dx^0\, \mathcal{E}$, so on $\mathbb{R}^4$ it is infinite, but

[3] Recall that the Bogomol'nyi equations for a monopole solution refer to the gauge $v_0 = 0$.

on $\mathbb{R}^3 \times S^1$ the finite energy of the monopoles means they have action $\frac{8\pi^2}{g^2}uR|k|$, where R is the radius of the circle.

The parameters X and Ω which appear in the one monopole solution are the bosonic collective coordinates of this configuration, relating to symmetry under translations and unbroken $U(1)$ gauge rotations, respectively. In general any monopole has $4|k|$ bosonic zero modes and, as shown by the Callias index theorem [56], $2|k|$ fermionic zero modes. The VEV parameter u breaks conformal invariance, but supersymmetry is still valid and two of the fermionic zero modes always correspond to that symmetry,

$$\lambda_\alpha^{\mathsf{susy}} = \sigma^{mn}{}_\alpha{}^\beta \xi_\beta v_{mn}. \tag{4.29}$$

Note that this is all of the fermionic zero modes for a one monopole configuration.

4.3.1 The extra monopole on the cylinder

The BPS solution is the only one monopole configuration on the cylinder that is independent of the periodic coordinate. There is another monopole on $\mathbb{R}^3 \times S^1$, however, which exists because of the S^1 part of the cylinder, and cannot be inherited from an (infinite action) $\mathbb{R}^4$ solution. In order to find it, we must consider the implications of modifying the space to $\mathbb{R}^3 \times S^1$ on the gauge group.

Recall that, in order to move to the cylinder, we imposed periodic boundary conditions on the gluon and gluino fields,

$$v_m(x^0, x^\mu) = v_m(x^0 + 2\pi R, x^\mu), \quad \lambda_\alpha(x^0, x^\mu) = \lambda_\alpha(x^0 + 2\pi R, x^\mu). \tag{4.30}$$

A gauge transformation U acts on v_m and λ as

$$v_m \mapsto U v_m U^{-1} + iU\partial_m U^{-1}, \tag{4.31}$$

$$\lambda_\alpha \mapsto U \lambda_\alpha U^{-1}, \tag{4.32}$$

so clearly any periodic gauge transformation, obeying $U(x^0, x^\mu) = U(x^0 + 2\pi R, x^\mu)$, will preserve the boundary conditions. However, we can more generally allow transformations with the property

$$U(x^0, x^\mu) = V \cdot U(x^0 + 2\pi R, x^\mu), \tag{4.33}$$

where V is an element of the centre of the gauge group, without disturbing equation (4.30). The centre is the subgroup of transformations that commute with all other elements of the group; for $SU(N)$ it is the $\mathbb{Z}_N$ group of N-th roots of unity $\{e^{\frac{2\pi i n}{N}}|n = 1, \ldots, N\}$ multiplying the unit matrix. With gauge group $SU(2)$ the centre is therefore composed of the unit matrix and minus the unit matrix, and an example of a non-periodic gauge transformation, which nevertheless leaves the fields periodic, is [57]

$$U_{\text{special}} = \exp\left(\frac{ix^0\tau_3}{2R}\right). \tag{4.34}$$

If we apply this transformation to a BPS one monopole in the singular gauge, then the VEV is changed to

$$\langle v_0 \rangle = -\left(u - \frac{1}{R}\right)\frac{\tau_3}{2}. \tag{4.35}$$

The VEV parameter u doesn't appear in the action in the BPS limit, so we can choose to define the theory with any particular value, but this argument shows that the value u is gauge equivalent to $u - \frac{1}{R}$. Therefore the set of theories on $\mathbb{R}^3 \times S^1$ possible at the classical level, or the classical moduli space, is isomorphic to a circle.

Once we have specified a value for u, all semiclassical configurations must obey the appropriate boundary condition, say, choosing a singular gauge,

$$v_0 \to \langle v_0 \rangle = -u\frac{\tau_3}{2}, \tag{4.36}$$

as $|x| \to \infty$. The configuration above, a BPS one monopole acted on by U_{special}, is not suitable, but if our starting point was a BPS one monopole with a modified VEV parameter[4] of $-u + \frac{1}{R}$, we would get a configuration with the right boundary condition. This can be used as the basis of a semiclassical calculation, and is the additional monopole that exists due to the cylinder. In [3] it is referred to as the affine monopole, but for now we shall follow [1] and call it the KK monopole. We already know the action of this monopole, because the action is gauge invariant and so will be equal to that of a BPS one monopole with the modified VEV parameter, namely $\frac{8\pi^2}{g^2}(1 - uR)$. The zero mode structure of the KK monopole also follows straight from that of the BPS monopole, in particular it has two supersymmetric adjoint fermion zero modes.

[4] If u is in the range $[0, 2\pi R)$ then so is $-u + \frac{1}{R}$. The transformation U_{special} maps this to $-u$, but the sign of the VEV parameter is irrelevant.

This is an unusual method for identifying a new semiclassical configuration, as normally a gauge transformation of a known solution would give an equivalent solution, rather than a distinct one. The independence of the KK monopole from the BPS monopole comes from the fact that when we change the space to $\mathbb{R}^3 \times S^1$ we must define the path integral on the cylinder. The main part of this is to define the measure, including which values of the gauge fields should be integrated over and which are equivalent; we only wish to integrate over gauge equivalence classes. The only straightforward way to do this is to sum over configurations that are inequivalent under the group of *periodic* gauge transformations, which is simply connected. Then the KK monopole must be considered to be from a different topological sector, and its contributions included by counting it as an entirely unrelated configuration.

4.3.2 Calorons

Caloron solutions were first found by Harrington and Shepard [58] (see also [49]), who sought solutions in non-zero temperature field theory analogous to Yang-Mills instantons. They are (anti-)self-dual and have integral Pontryagin index, despite the fact that it need not be quantised on $\mathbb{R}^3 \times S^1$. Furthermore, as the radius of the circle tends to infinity, any caloron solution approaches the form of a four dimensional instanton. However, these are solutions in which all components of the gauge field, including the zeroth, v_0, fall to zero at infinity. As we saw in section 4.3.1, though, we can define the theory on $\mathbb{R}^3 \times S^1$ with any value of $\langle v_0 \rangle$, or equivalently any asymptotic value of the Wilson loop,

$$\lim_{|x|\to\infty} \left(\int_0^{2\pi R} dx^0 \, v_0 \right) = 2\pi R \langle v_0 \rangle. \tag{4.37}$$

This is not relevant to non-zero temperature field theory because, in that case, effects from configurations with a non-zero Wilson loop are suppressed [49]. Caloron solutions with non-trivial Wilson loop were constructed and studied only recently. Intriguingly, the caloron with Pontryagin index unity can be shown to be made up of one BPS monopole and one KK monopole [57, 59, 60]. Note that the actions of those monopoles show that they have Pontryagin indices of uR and $1 - uR$ respectively, where u lies between 0 and $\frac{1}{R}$, so the sum is one as required. Also, the gauge transformation

U_{special} reverses the sign of the magnetic charge, so the KK monopole has $q = -1$ and the caloron has zero net magnetic charge, as it should. Note, however, that the KK monopole is not an anti-monopole, crucially because it has zero modes for λ but not for $\overline{\lambda}$.

Therefore, monopoles act as the partons of calorons in the case of a non-trivial Wilson loop[5]. This does not help us to interpret the meron configurations on $\mathbb{R}^4$, but the circle acts as a regulator to remove the pathological nature of those solutions and replace it with the smooth behaviour of the monopoles.

4.3.3 Monopoles and calorons in any gauge group

The entire picture of monopoles and calorons, and the relationships between them, which we have described above for $SU(2)$ theories, can be generalised to the case of any other gauge group. We shall discuss the necessary changes here; useful definitions and terminology can be found in appendix B.

In general, for any gauge group, we may use gauge transformations to rotate the VEV so that it lies in the Cartan subalgebra,

$$\langle v_0 \rangle = -V_i H^i. \tag{4.38}$$

This shows that the gauge group is spontaneously broken to its maximal abelian subgroup[6], $U(1)^n$, where n is the rank. Classical solutions can then possess magnetic charges associated with each of the $U(1)$ groups, as well as the Pontryagin index.

All one monopole solutions in an arbitrary gauge group are embeddings of $SU(2)$ solutions in appropriate subgroups, analogously to one instantons in $\mathbb{R}^4$. In contrast, however, because of the distinct magnetic charges, monopoles embedded in different $SU(2)$ subgroups are examples of different types of monopoles. Therefore, one monopoles in any gauge group have the same numbers of zero modes as one monopoles in $SU(2)$, that is four bosonic zero modes (three translational and one from the relevant unbroken $U(1)$ subgroup) and two (supersymmetric) adjoint fermion zero modes.

[5]If the Wilson loop or VEV vanishes, the monopoles become trivial and cannot be semiclassical configurations or constituents of other configurations.

[6]It is possible to arrange for there to be an unbroken non-abelian subgroup, in the special circumstance that $\alpha \cdot V \equiv \alpha^i V_i = 0$ for some root α, but we shall always assume that this is not the case.

Each $SU(2)$ subgroup of a simple Lie group is associated with a root α, and the corresponding generators are

$$J_1 = \frac{1}{\sqrt{2L}}(E_\alpha + E_{-\alpha}), \quad J_2 = \frac{1}{\sqrt{2L}i}(E_\alpha - E_{-\alpha}), \quad J_3 = \frac{1}{L}\alpha^* \cdot H, \tag{4.39}$$

which obey

$$[J_a, J_b] = i\epsilon_{abc}J_c. \tag{4.40}$$

An $SU(2)$ BPS monopole solution can then be embedded as

$$v_\mu = w^c_\mu J_c, \tag{4.41}$$

$$v_0 = \Phi^c J_c - \left(V - \frac{1}{L}(\alpha \cdot V)\alpha^*\right) \cdot H, \tag{4.42}$$

where

$$w^c_\mu = \epsilon_{\mu\nu c}\frac{x^\nu}{|x|^2}\left(1 - \frac{u|x|}{\sinh u|x|}\right), \tag{4.43}$$

$$\Phi^c = -\frac{x^c}{|x|^2}\left(u|x| \coth u|x| - 1\right), \tag{4.44}$$

for a monopole located at the origin and without a $U(1)$ rotation, and $u = \alpha \cdot V$. Note that for gauge group $SU(2)$ itself, which has rank $n = 1$, the Cartan subalgebra generator can be taken to be[7] $H = \frac{1}{\sqrt{2}}\tau_3$, and the positive root is $\alpha = \sqrt{2}$, so $\alpha \cdot V$ coincides with the parameter u in the previous sections of this chapter.

Analysis of the zero modes of these solutions [61], using the Callias index theorem, shows that only some of these embeddings are true one monopole solutions, with four bosonic zero modes and two adjoint fermion zero modes. The rest are configurations that are particular combinations of these fundamental monopoles. In order to state what the fundamental BPS monopoles are, we first note that any choice of the VEV in the form given in equation (4.38) provides a natural ordering[8] for the roots, in that a root may be considered positive or negative according to the sign of $\alpha^i V_i$. The BPS one monopoles of any gauge group are then the ones where the $SU(2)$ BPS monopole

[7] Note the normalisation convention in equation (B.5).

[8] Any root ordering method of the type discussed in appendix B is arbitrary but easily related to the other consistent choices by gauge transformations. Similarly, starting from a VEV like that in equation (4.38), one can gauge transform to other possibilities without taking the VEV out of the Cartan subalgebra. In fact, permutations of the elements $\{V_i\}$, achievable via gauge transformations, are in one-to-one correspondence with the Weyl reflections on the roots that relate different orderings.

is embedded in the $SU(2)$ subgroup of a root that is *simple* according to this ordering. There are always n simple roots, so there are n types of fundamental BPS monopoles, which agrees with the n types of magnetic charge coming from the unbroken subgroup $U(1)^n$. The BPS one monopole associated with the simple root $\alpha_{(i)}$ has action $\frac{8\pi^2 R}{g^2}\alpha^*_{(i)} \cdot V$.

For gauge groups other than $SU(2)$, we have non-periodic gauge transformations of the form $U_\omega = \exp\left(\frac{ix^0}{R}\omega_* \cdot H\right)$, where ω_* is a coweight. These map V_i to $V_i - \frac{1}{R}\omega_{*i}$ (hence $\alpha^i V_i$ is shifted by a multiple of $\frac{1}{R}$), so the classical moduli space is isomorphic to $\frac{\mathbb{R}^n}{\Lambda^*_W \rtimes W}$, where Λ^*_W is the lattice of linear combinations of coweights with integer coefficients, and W is the Weyl group.

We can use the non-periodic transformations $U_\alpha = \exp\left(\frac{ix^0}{R}J_3(\alpha)\right)$, applied to a monopole associated with root α, but with modified VEV parameter of $-\alpha \cdot V + \frac{1}{R}$, to obtain the various extra monopoles on the cylinder. However, while all configurations constructed in this way are solutions, only the one starting from the *lowest* root, $\alpha_{(0)} \equiv -\theta$, in the natural ordering defined above, is a fundamental KK or affine monopole [59]. It has action $\frac{8\pi^2}{g^2}\left(1 - R\alpha^*_{(0)} \cdot V\right)$.

It has also been shown [59] that the caloron solution in any gauge group is a combination of a BPS monopole solution with winding number m^i for each simple root $\alpha_{(i)}$ (where the $\{m^i\}$ are the comarks) and a single KK monopole. The number of bosonic zero modes, for example, is then

$$4\left(1 + \sum_{i=1}^{n} m^i\right) = 4c_2, \tag{4.45}$$

in agreement with our expectations for a caloron or one instanton. In addition, with this combination of monopoles all the magnetic charges cancel to give a neutral configuration.

4.4 Methodology

To summarize, the idea behind the calculations in this thesis is to work in a theory defined on $\mathbb{R}^3 \times S^1$, with a non-trivial VEV. Classically the VEV can take any value, but quantum corrections can lift this degeneracy, as will be seen in chapter 5, and the quantum moduli space is a single point with a particular value of u, rather than, say, a circle for $SU(2)$. In such a theory, the semiclassical configurations are all possible combinations of $n+1$ types of fundamental monopoles, which are the partons of the configurations analogous to instantons in $\mathbb{R}^4$. The fundamental monopoles all have two adjoint fermion zero modes, and so the gluino condensate in this theory can be calculated as the sum of their direct contributions.

The quantum value of the VEV parameter u is proportional to $\frac{1}{R}$ (and classically it is gauge equivalent to a value bounded by this amount, by periodicity), so if R is chosen to be small, then u will be large. In this limit, the coupling is small, as can be seen from equation (5.53) with μ chosen to be equal to the scale $u \gg \Lambda$, so semiclassical calculations are justified and of the WCI type. On the other hand, the answer for any Green's function should be holomorphic in u, by supersymmetry, and will therefore also be holomorphic in R, allowing us to analytically continue the result to $R \to \infty$ and four large dimensions.

A similar strategy, employed previously in [62], is to consider the theory defined on $T^4 = (S^1)^4$, so that all the directions are periodic. In this case the fundamental configurations are solutions called torons [63], which also have two adjoint fermion zero modes meaning that the gluino condensate can be calculated directly. However, the four periods have to be fine-tuned in order to maintain the existence of the finite action torons, and because there is more than one period it is no longer possible to rely on the holomorphy of Green's functions on those parameters. This makes attempting to find an unambiguous $\mathbb{R}^4$ limit severely difficult, and consequently it is perhaps not surprising that the value of the gluino condensate on the four torus calculated via torons does not coincide with either the SCI or WCI results in $\mathbb{R}^4$.

In the next chapter we shall apply the monopole method to the calculation of the gluino condensate in supersymmetric pure Yang-Mills theory will gauge group $SU(2)$, in chapter 6 we will extend the analysis to any gauge group, and also consider the inclusion of matter fields.

Chapter 5

One monopole calculations

In this chapter we shall calculate the gluino condensate in $\mathcal{N} = 1$ supersymmetric pure Yang-Mills theory, using monopoles on $\mathbb{R}^3 \times S^1$, in order to demonstrate how to find one monopole contributions to correlation functions, and to compare with the SCI and WCI results. The actual evaluation is not too arduous, but before we can proceed with it we need to establish that our one monopole configurations correspond to a true quantum vacuum.

The classical moduli space of supersymmetric pure $SU(2)$ gauge theory on $\mathbb{R}^3 \times S^1$ is given by the set of distinct values that the VEV, or equivalently the Wilson loop at spatial infinity, may take. This is a circle, as was discussed in the section 4.3.1. However, quantum effects can, and indeed do, lift the classical degeneracy; not all classical vacua remain ground states of the full quantum theory.

The search for the quantum vacuum will occupy the majority of this chapter, as it also requires a one monopole calculation, in order to find the low energy effective action on the cylinder.

5.1 Determination of the quantum vacuum

To discover how quantum effects may alter a classical vacuum state, it is useful to consider the low energy dynamics. In particular, we can attempt to calculate the Wilsonian effective action [64, 65], which is the action of the remaining degrees of freedom after massive fields and fields with virtuality greater than some scale M have been integrated out. Since it is an effective action, valid only below the scale M, it need not be renormalisable, and so will in general be more complicated than the microscopic description of the theory. Nevertheless, the lowest energy state of the Wilsonian effective action will be the same as the vacuum of the full theory, and it may be easier to identify.

The integration is not carried out directly. Instead, having defined the Wilsonian effective action in this way, the normal procedure is to find the most general form it can take subject to the symmetries it must respect, and then determine as much of it as possible by finding all parts that are susceptible to direct calculation.

5.1.1 Classical low energy dynamics

The appropriate variables for the Wilsonian effective action are the classically massless degrees of freedom. Here we shall identify these variables, and consider their low energy dynamics *before* quantum corrections are taken into account.

The non-vanishing VEV,

$$\langle v_0 \rangle = -u\frac{\tau_3}{2}, \tag{5.1}$$

spontaneously breaks the gauge group $SU(2)$ to $U(1)$. For any field ϕ that transforms in the adjoint representation, we can write

$$\phi = \phi^a \frac{\tau_a}{2}. \tag{5.2}$$

The components ϕ^1 and ϕ^2 gain mass u under the Higgs mechanism, but ϕ^3, which describes the part of ϕ parallel to the VEV, remains massless at the classical level.

In addition, all fields must be periodic in the compact direction, so they can be expressed as a Fourier series,

$$\phi = \sum_{n=-\infty}^{n=+\infty} \phi_n e^{in\frac{x_0}{R}}. \tag{5.3}$$

Then, the kinetic term in the action for a bosonic field contains

$$\int_0^{2\pi R} dx_0 \left(\partial_0 \phi^\dagger \partial_0 \phi\right) = \sum_{n=-\infty}^{n=+\infty} 2\pi R \left(\frac{n}{R}\right)^2 |\phi_n|^2, \tag{5.4}$$

so all Fourier components have a Kaluza-Klein mass, *except* for the one with $n = 0$. The part of ϕ which has no dependence on x_0 is classically massless.

Therefore when constructing the Wilsonian effective action, we must only keep the x_0-independent parts of fields parallel to the VEV. When we write v_m or λ in the remainder of this section (5.1), we shall mean exactly those components, with $a = 3$ and $n = 0$.

The classical low energy dynamics is an abelian[1] gauge theory in three dimensions, with action

$$S_0 = \frac{2\pi R}{g^2}\int d^3x \left(\frac{1}{2}\frac{1}{(2\pi R)^2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{4}v_{\mu\nu}v^{\mu\nu} + i\overline{\lambda}\overline{\sigma}^\mu\partial_\mu\lambda\right), \tag{5.5}$$

where $v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$ and we use the Wilson loop $\varphi = \int dx_0\, v_0 = 2\pi R v_0$. The descendant of a theta term from the full microscopic theory is

$$S_\vartheta = -\frac{i\vartheta}{16\pi^2}\int d^3x\, \epsilon^{\mu\nu\rho}v_{\nu\rho}\partial_\mu\varphi. \tag{5.6}$$

Before we calculate the quantum corrections to this action, we shall transform it to a more convenient but equivalent (dual) form.

5.1.2 The dual theory

In the low energy regime under consideration, our theory is effectively three dimensional, and we have the possibility of including a three dimensional analogue of a four dimensional theta term. This contribution would take the form

$$S_\sigma = \frac{i\sigma}{8\pi}\int d^3x\, \epsilon^{\mu\nu\rho}\partial_\mu v_{\nu\rho}. \tag{5.7}$$

As in the four dimensional case, this is a topological term, which means that it is the integral of a total derivative. Also similarly, it may be evaluated in terms of a topological charge - in this case, the magnetic charge, as is easily seen from equation (5.7). Therefore, just as in Yang-Mills-Higgs theory, we have

$$\frac{1}{8\pi}\int d^3x\, \epsilon^{\mu\nu\rho}\partial_\mu v_{\nu\rho} = n \in \mathbb{Z}, \tag{5.8}$$

which is consistent with Dirac quantisation of magnetic charge [50]. Note that in the path integral, this sigma term will give

$$e^{-S_\sigma} = e^{-in\sigma}, \tag{5.9}$$

[1]This corresponds to the unbroken $U(1)$ subgroup.

Therefore when constructing the Wilsonian effective action, we must only keep the x_0-independent parts of fields parallel to the VEV. When we write v_m or λ in the remainder of this section (5.1), we shall mean exactly those components, with $a = 3$ and $n = 0$.

The classical low energy dynamics is an abelian[1] gauge theory in three dimensions, with action

$$S_0 = \frac{2\pi R}{g^2} \int d^3x \left(\frac{1}{2} \frac{1}{(2\pi R)^2} \partial_\mu \varphi \partial^\mu \varphi + \frac{1}{4} v_{\mu\nu} v^{\mu\nu} + i\bar{\lambda}\bar{\sigma}^\mu \partial_\mu \lambda \right), \tag{5.5}$$

where $v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$ and we use the Wilson loop $\varphi = \int dx_0\, v_0 = 2\pi R v_0$. The descendant of a theta term from the full microscopic theory is

$$S_\vartheta = -\frac{i\vartheta}{16\pi^2} \int d^3x\, \epsilon^{\mu\nu\rho} v_{\nu\rho} \partial_\mu \varphi. \tag{5.6}$$

Before we calculate the quantum corrections to this action, we shall transform it to a more convenient but equivalent (dual) form.

5.1.2 The dual theory

In the low energy regime under consideration, our theory is effectively three dimensional, and we have the possibility of including a three dimensional analogue of a four dimensional theta term. This contribution would take the form

$$S_\sigma = \frac{i\sigma}{8\pi} \int d^3x\, \epsilon^{\mu\nu\rho} \partial_\mu v_{\nu\rho}. \tag{5.7}$$

As in the four dimensional case, this is a topological term, which means that it is the integral of a total derivative. Also similarly, it may be evaluated in terms of a topological charge - in this case, the magnetic charge, as is easily seen from equation (5.7). Therefore, just as in Yang-Mills-Higgs theory, we have

$$\frac{1}{8\pi} \int d^3x\, \epsilon^{\mu\nu\rho} \partial_\mu v_{\nu\rho} = n \in \mathbb{Z}, \tag{5.8}$$

which is consistent with Dirac quantisation of magnetic charge [50]. Note that in the path integral, this sigma term will give

$$e^{-S_\sigma} = e^{-in\sigma}, \tag{5.9}$$

[1]This corresponds to the unbroken $U(1)$ subgroup.

so σ is an angular variable, because adding multiples of 2π to it does not affect the path integral.

It is useful to change variables using a modification of the sigma term. This will correspond to finding a dual description of the same effective theory which is simpler to work with. Initially we make σ into an auxiliary field, depending on the spatial coordinates and integrated over in the path integral, but without a kinetic term in the action. It then acts as a Lagrange multiplier field, imposing the Bianchi identity at every point in space, through

$$S_{\text{constraint}} = \frac{i}{8\pi} \int d^3x \, \sigma \epsilon^{\mu\nu\rho} \partial_\mu v_{\nu\rho}, \tag{5.10}$$

and (incorporating multiplicative factors into the measure),

$$\int d\varphi \, dv_\mu \, d\lambda \, d\sigma \, e^{-S_0 - S_\vartheta - S_{\text{constraint}}} = \int d\varphi \, dv_\mu \, d\lambda \, e^{-S_0 - S_\vartheta} \delta \left(\epsilon^{\mu\nu\rho} \partial_\mu v_{\nu\rho} \right). \tag{5.11}$$

The advantage of using the gauge potential as the fundamental variable is that the Bianchi identity follows automatically from the definition of the field strength. However, with the Bianchi identity artificially enforced, we are free to use the field strength itself as the variable[2]. We shall therefore formally change variables in the path integral,

$$\int d\varphi \, dv_\mu \, d\lambda \, d\sigma \, e^{-S[\varphi, v_\mu, \lambda, \sigma]} = \int d\varphi \, dB_\mu \, d\lambda \, d\sigma \, e^{-S[\varphi, B_\mu, \lambda, \sigma]}, \tag{5.12}$$

where $S = S_0 + S_\vartheta + S_{\text{constraint}}$ and we use $B_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho} v_{\nu\rho}$. If we integrate $S_{\text{constraint}}$ by parts,

$$S_{\text{constraint}} = -\frac{i}{8\pi} \int d^3x \, \epsilon^{\mu\nu\rho} v_{\nu\rho} \partial_\mu \sigma + in\langle\sigma\rangle, \tag{5.13}$$

with $in\langle\sigma\rangle = in \lim_{|x|\to\infty} \sigma$ a constant we can neglect, then the action S depends quadratically on B_μ,

$$S = \int d^3x \, \left(a B_\mu B^\mu + b_\mu B^\mu + c \right), \tag{5.14}$$

[2]Note that this is true only in a $U(1)$ theory; in the non-abelian case not all the information about the gauge field is contained in the field strength.

where

$$a = \frac{\pi R}{g^2}, \tag{5.15}$$

$$b_\mu = -\frac{i}{4\pi}\partial_\mu\left(\sigma + \frac{\vartheta}{2\pi}\varphi\right), \tag{5.16}$$

$$c = \frac{1}{2\pi R}\left(\frac{1}{2g^2}\partial_\mu\varphi\partial^\mu\varphi + \left(\frac{2\pi R}{g}\right)^2 i\bar{\lambda}\bar{\sigma}^\mu\partial_\mu\lambda\right). \tag{5.17}$$

Therefore we can completely integrate out B_μ using Gaussian integration. First we complete the square,

$$aB_\mu B^\mu + b_\mu B^\mu + c = a\left(B_\mu + \frac{1}{2a}b_\mu\right)^2 - \frac{1}{4a}b_\mu b^\mu + c, \tag{5.18}$$

then shift $B_\mu \mapsto B'_\mu = B_\mu + \frac{1}{2a}b_\mu$, and finally integrate,

$$\int d\varphi\, dB_\mu\, d\lambda\, d\sigma\, e^{-S[\varphi,B_\mu,\lambda,\sigma]} = \int d\varphi\, dB'_\mu\, d\lambda\, d\sigma\, e^{-S[\varphi,B'_\mu,\lambda,\sigma]} \tag{5.19}$$

$$= \int d\varphi\, d\lambda\, d\sigma\, e^{-S_{\text{dual}}[\varphi,\lambda,\sigma]}, \tag{5.20}$$

where $S_{\text{dual}} = \int d^3x\, \left(-\frac{1}{4a}b_\mu b^\mu + c\right)$, or

$$\begin{aligned} S_{\text{dual}} =& \frac{1}{2\pi R}\int d^3x\left(\frac{1}{2g^2}\partial_\mu\varphi\partial^\mu\varphi + \left(\frac{2\pi R}{g}\right)^2 i\bar{\lambda}\bar{\sigma}^\mu\partial_\mu\lambda\right) \\ &+ \frac{1}{2\pi R}\int d^3x\,\frac{g^2}{32\pi^2}\left(\partial_\mu\sigma + \frac{\vartheta}{2\pi}\partial_\mu\varphi\right)^2. \end{aligned} \tag{5.21}$$

We have now completed the change of variables, and arrived at the dual description of the classical low energy theory. We have eliminated v_μ, and in favour of it σ has become a dynamical field, with a kinetic term in S_{dual}. When σ was first introduced, it was coupled to the magnetic charge, so we may think of it as a magnetic photon field, dual to the electric photon v_μ. However, σ is a scalar, due to the fact that the Bianchi identity is a scalar equation in three dimensions. Our theory now contains only scalar fields and fermions, which is a great simplification compared to the original vector theory[3].

5.1.3 The general form of the effective action

The classical effective action we have been considering has $\mathcal{N} = 2$ three dimensional supersymmetry inherited from the $\mathcal{N} = 1$ four dimensional supersymmetry. This is

[3]This technique would not be of benefit in four dimensions, as in that case the Bianchi identity has four components, and in moving to the dual theory we would replace one vector field with another.

most easily seen from the dual form of the action. If we combine the real scalars φ and σ into a complex field[4],

$$z = -i(\tau\varphi + \sigma), \tag{5.22}$$

where

$$\tau = \frac{4\pi i}{g^2} + \frac{\vartheta}{2\pi} \tag{5.23}$$

is the complexified coupling, and define a fermionic field with a different normalisation,

$$\psi = \frac{2^{\frac{7}{2}}\pi^2 R}{g^2}\lambda, \tag{5.24}$$

then the classical effective action becomes

$$S_{\text{dual}} = \frac{1}{16\pi^2 R}\int d^3x \, \frac{1}{\operatorname{Im}\tau}\left(\partial_\mu z^\dagger \partial^\mu z + i\overline{\psi}\overline{\sigma}^\mu \partial_\mu \psi\right). \tag{5.25}$$

This is invariant under the supersymmetry transformations

$$\delta z = \sqrt{2}\xi\psi, \tag{5.26}$$

$$\delta\psi = \sqrt{2}i\sigma^\mu \overline{\xi}\partial_\mu z, \tag{5.27}$$

with an infinitesimal transformation parameter ξ. The fields z and ψ, and an auxiliary field $\mathcal{F}$ with classical equation of motion $\mathcal{F} = 0$, together form a complex chiral superfield,

$$X = z + \sqrt{2}\theta\psi + \theta\theta\mathcal{F}, \tag{5.28}$$

that represents two interlinked real chiral superfields, and in terms of which the classical effective action can be written

$$S_{\text{dual}} = \frac{1}{16\pi^2 R}\int d^3x \, d^2\theta \, d^2\overline{\theta} \, \frac{1}{\operatorname{Im}\tau} X^\dagger X. \tag{5.29}$$

Now we consider the quantum version of this action. There is nothing in the theory that explicitly or spontaneously breaks supersymmetry, so the action must respect this symmetry in all regimes including low energy. The most general supersymmetric action of a complex chiral superfield, containing no more than two derivatives [65, 39] is

$$S_{\text{susy}} = \int d^3x \left(\int d^2\theta \, d^2\overline{\theta} \, \mathcal{K}(X, X^\dagger) + \int d^2\theta \, \mathcal{W}(X) + \int d^2\overline{\theta} \, \mathcal{W}^\dagger(X^\dagger)\right), \tag{5.30}$$

[4]Notice that both φ and σ are 2π periodic.

where $\mathcal{W}$ and $\mathcal{W}^\dagger$ are required to be holomorphic and anti-holomorphic respectively. The Wilsonian effective action must therefore have this form.

The function $\mathcal{W}$ is called the *superpotential*, and it appears in an F-term in the action, which implies that it will be zero to all orders in perturbation theory, but can have non-vanishing, exactly calculable, monopole contributions, as discussed in section 3.3. In contrast, the D-term involving $\mathcal{K}$ will be dominated by incalculable perturbative effects. Fortunately, it is the superpotential that holds information about the vacuum, as we shall now show.

If the bosonic part of the action is expanded in the bosonic components of X, the auxiliary field $\mathcal{F}$ and its conjugate are present only in the following terms,

$$\int d^3x \left(-G\mathcal{F}^\dagger\mathcal{F} + \frac{\partial\mathcal{W}(z)}{\partial z}\mathcal{F} + \frac{\partial\mathcal{W}^\dagger(z^\dagger)}{\partial z^\dagger}\mathcal{F}^\dagger\right), \tag{5.31}$$

with $G = \frac{\partial^2\mathcal{K}(z,z^\dagger)}{\partial z\partial z^\dagger}$. The Euler-Lagrange equation for $\mathcal{F}$ then gives

$$\mathcal{F}^\dagger = \frac{1}{G}\frac{\partial\mathcal{W}(z)}{\partial z}, \tag{5.32}$$

and substituting this and its conjugate back into the action, we find the potential

$$\int d^3x\, V = \int d^3x\, \frac{1}{G}\frac{\partial\mathcal{W}^\dagger(z^\dagger)}{\partial z^\dagger}\frac{\partial\mathcal{W}(z)}{\partial z}. \tag{5.33}$$

This is positive semi-definite, as we should expect since there is a general result in supersymmetric theories that the Hamiltonian must be positive semi-definite (see section 3.2.3, and [20]). It also follows from the same general arguments that only states which have an expectation value of zero for the energy are supersymmetric. So, if we can solve what is called the F-flatness condition,

$$\frac{\partial\mathcal{W}(z)}{\partial z} = 0, \tag{5.34}$$

to find a supersymmetric state, this will also give an absolute minimum of the energy and hence a quantum vacuum state[5]. Therefore, our aim is to determine the superpotential.

[5]We took $G \neq 0$ above, which was an assumption as we do not know the form of $\mathcal{K}$. However, if $G = 0$ then the F-flatness condition follows directly from the Euler-Lagrange equations.

5.1.4 Calculation of the superpotential

The strategy we shall employ to find the superpotential is to calculate a term in its expansion and then use holomorphy to reconstruct the whole. The F-term in the effective action can be written as

$$\int d^3x \int d^2\theta\, \mathcal{W}(X) = \int d^3x \left(\frac{\partial \mathcal{W}(z)}{\partial z} \mathcal{F} + \frac{1}{2} \frac{\partial^2 \mathcal{W}(z)}{\partial z^2} \psi\psi \right). \tag{5.35}$$

When z has a VEV, the second term includes

$$\int d^3x \, \frac{1}{2} \left\langle \frac{\partial^2 \mathcal{W}(z)}{\partial z^2} \right\rangle \psi\psi, \tag{5.36}$$

which is an effective mass term for the fermion ψ, or after rescaling, for λ. Recall however, that λ is the part of the gluino field that is classically massless. Therefore it is quantum effects, via the superpotential, that generate a mass for this component. This is the mass term that we shall evaluate, using semiclassical techniques, and from which we can infer the superpotential.

We shall approach this by investigating the correlator

$$\langle \lambda_\alpha(x) \lambda_\beta(0) \rangle \,. \tag{5.37}$$

We can use an LSZ reduction formula to relate this function to a scattering amplitude, and then, in the limit of large $|x|$, it can be shown that the asymptotic form is two massless propagators multiplied by a constant including the mass of λ, or in configuration space language,

$$\lim_{|x| \to \infty} \langle \lambda_\alpha(x) \lambda_\beta(0) \rangle = \left(\frac{g^2}{2\pi R} \right)^2 m_\lambda \int d^3x' \, \left(\mathcal{S}_{\mathrm{F}}(x - x') \mathcal{S}_{\mathrm{F}}(-x') \right)_{\alpha\beta} . \tag{5.38}$$

Note that we use m_λ to mean the coefficient multiplying $\frac{1}{2}\lambda\lambda$ in the action, which is not the physical mass because λ is not a canonically normalised field (see the kinetic term in equation (5.5)). Similarly, we are attempting to find the coefficient in front of $\frac{1}{2}\psi\psi$ and will not be concerned with the actual value of the mass. Another effect of this normalisation is that the constant in front of the integral above is not simply the mass. We have also defined $\mathcal{S}_{\mathrm{F}}(x) = \frac{\sigma^\mu x_\mu}{4\pi |x|^3}$, the massless fermion propagator in three dimensional configuration space, or equivalently the Green's function of the Dirac operator $i\bar{\sigma}^\mu \partial_\mu$.

Clearly only (multi-)monopole solutions with two fermionic zero modes will affect the value of the correlator, which implies that just the two fundamental monopoles will contribute. In order to perform the semiclassical calculations around these configurations, we need to know the one monopole measure and the adjoint fermion zero modes.

The latter are given by the supersymmetric zero modes (see section 4.3), which all multimonopole solutions have by virtue of supersymmetry. In the case of the fundamental monopoles they are the *only* two zero modes, with two Grassmannian collective coordinates ξ_1 and ξ_2. In fact, although we know the field strength and hence the zero modes completely for the one monopole solution, at this point we shall only require knowledge of the asymptotic form of the zero modes,

$$\lim_{|x|\to\infty} \lambda_\alpha(x) = \lambda_\alpha^{\mathrm{LD}}(x) = 8\pi(\mathcal{S}_{\mathrm{F}}(x-X)\xi)_\alpha. \tag{5.39}$$

The measure is shown in appendix C to be

$$\int d\mu_{1-\mathrm{mono}} = \frac{\mu^3 R}{g^2} e^{-S} \int d^3X \int_0^{2\pi} d\Omega \int d^2\xi, \tag{5.40}$$

with S the monopole action, μ the Pauli-Villars renormalisation scale, X the centre of the monopole, and Ω the $U(1)$ gauge orientation.

Now we can bring everything together and, integrating over the gauge and fermionic collective coordinates, find an expression for the contribution of either monopole to the correlator in the limit of large $|x|$,

$$\langle \lambda_\alpha(x)\lambda_\beta(0)\rangle_{\mathrm{BPS/KK}} \overset{|x|\to\infty}{\longrightarrow} \frac{\mu^3 R}{g^2} e^{-S} \int d^3X\, d\Omega\, d^2\xi\, \lambda_\alpha^{\mathrm{LD}}(x)\lambda_\beta^{\mathrm{LD}}(0) \tag{5.41}$$

$$= \frac{2^6\pi^3\mu^3 R}{g^2} e^{-S} \int d^3X\, (\mathcal{S}_{\mathrm{F}}(x-X)\mathcal{S}_{\mathrm{F}}(-X))_{\alpha\beta}. \tag{5.42}$$

We can amputate the propagators to isolate the coefficient in front of the integral, then find the mass term for λ, and (after applying the conversion factor, equation (5.24)) also for ψ, given by either monopole,

$$\int d^3x\, \frac{1}{2}\left[\frac{2^6\pi^3\mu^3 R}{g^2}\left(\frac{2\pi R}{g^2}\right)^2 e^{-S}\right]\lambda\lambda = \int d^3x\, \frac{1}{2}\left[\frac{2\pi\mu^3 R}{g^2} e^{-S}\right]\psi\psi. \tag{5.43}$$

The BPS monopole has action $-\langle z\rangle$ and the KK monopole has action $-2\pi i\tau + \langle z\rangle$, using the results given in chapter 4, and including the effect of the field σ. Summing over both contributions we find the total quantum generated mass term for ψ, from which, using equation (5.36), we can identify the VEV of the second derivative of the superpotential,

$$\left\langle \frac{\partial^2 \mathcal{W}(z)}{\partial z^2} \right\rangle = \frac{2\pi\mu^3 R}{g^2} \left(e^{\langle z \rangle} + e^{2\pi i\tau - \langle z \rangle} \right). \tag{5.44}$$

This represents only one term of the superpotential, given by single monopole contributions, and all other terms in the expansion are generated by analogous multimonopole effects. However, rather than calculate all these terms individually, we can realise that the holomorphic property of $\mathcal{W}$ constrains them, and that we must be able to reconstruct the full superpotential by promoting $\langle z \rangle$ to z, and then in turn z to X. Therefore we have found

$$\mathcal{W}(X) = \frac{2\pi\mu^3 R}{g^2} \left(e^{X} + e^{2\pi i\tau - X} \right). \tag{5.45}$$

We can now use this result to discover the quantum vacuum state, since

$$\frac{\partial \mathcal{W}(z)}{\partial z} = 0 \quad \Rightarrow \quad e^{2\langle z \rangle} = e^{2\pi i \tau}, \tag{5.46}$$

$$\Rightarrow \quad \langle z \rangle = \pi i(\tau + \nu), \tag{5.47}$$

with $\nu \in \mathbb{Z}$, and the VEV of z contains the VEV of the Wilson loop, $\langle \varphi \rangle = -\pi$, which parametrises the vacuum. The quantum corrections have lifted the degeneracy of the classical moduli space, leaving just one point of it as a true vacuum state.

The parameter ν can in principle take any integral value, but since $\langle z \rangle$ is 2π periodic, only the values 0 and 1 (say) are distinct. Shifting $\vartheta \mapsto \vartheta + 2\pi$ sends $\nu \mapsto \nu + 1$, so the vacua corresponding to these two values have the same physical properties. This is in accordance with Witten's index [40], which predicts two distinct but physically equivalent vacua in supersymmetric pure $SU(2)$ gauge theory.

5.1.5 Confinement

We can take $\langle \sigma \rangle$ to be zero, but due to the non-zero VEV of the Wilson loop we find a term like $\frac{1}{2} m_\sigma \sigma^2$ on expanding the superpotential about the vacuum, so σ is a massive field. This implies confinement of the electric degrees of freedom, through a dual Meissner effect. In superconductors the electric photon gains an effective mass and the magnetic fields are confined to the exterior of the superconductor or flux tubes. In the present case the words electric and magnetic should be exchanged, and all the remaining electric fields should be removed from the low energy spectrum. This effect was first noticed by Polyakov in the context of a genuinely three dimensional gauge theory, in [47].

5.2 Calculation of the gluino condensate in $SU(2)$

In section 5.1 we examined a reduced, low energy form of the theory in order to determine the quantum vacuum state. With this knowledge we can now return to the full theory and calculate the gluino condensate in the correct vacuum.

5.2.1 The semiclassical method

We can find $\langle \mathrm{Tr}\, \lambda\lambda \rangle$ using a direct semiclassical calculation. By design we have modified the theory such that we can identify and manipulate the configurations contributing to the gluino condensate with mathematical rigour. These are gauge configurations with two adjoint fermion zero modes, and are the one monopole solutions. Using the one monopole measure derived in appendix C, we can find the contribution of either monopole to the gluino condensate,

$$\langle \mathrm{Tr}\, \lambda\lambda(x) \rangle_{\mathrm{BPS/KK}} = \frac{\mu^3 R}{g^2} e^{-S} \int d^3X \int_0^{2\pi} d\Omega \int d^2\xi \, \mathrm{Tr}\left[\lambda^{\mathrm{ss}}\lambda^{\mathrm{ss}}(x-X)\right]. \tag{5.48}$$

The classical values of the gluino fields are given by the supersymmetric zero modes,

$$\lambda^{\mathrm{ss}}_\alpha = \sigma^{mn}{}_\alpha{}^\beta \xi_\beta v^{\mathrm{class}}_{mn}, \tag{5.49}$$

with v^{class}_{mn} the field strength of the monopole configuration. Applying equation (A.14) and $v^{\mathrm{class}}_{mn} = *v^{\mathrm{class}}_{mn}$, and integrating over ξ and Ω, we find

$$\langle \mathrm{Tr}\, \lambda\lambda(x) \rangle_{\mathrm{BPS/KK}} = \frac{2\pi\mu^3 R}{g^2} e^{-S} \int d^3X \, \mathrm{Tr}\left[v^{\mathrm{class}}_{mn} v^{\mathrm{class}}_{mn}(x-X)\right]. \tag{5.50}$$

We can change variables from X to $x - X$, and then the integral is clearly just proportional to the real part of the monopole action[6], so we can immediately evaluate it to

[6]That is, not including the contribution of the theta term

give

$$\langle \mathrm{Tr}\,\lambda\lambda \rangle_{\mathrm{BPS/KK}} = \frac{2\pi\mu^3 R}{g^2} e^{-S} \frac{g^2}{\pi R} \mathrm{Re}(S) = 2\mu^3 \,\mathrm{Re}(S)\, e^{-S}. \tag{5.51}$$

Recall from section 5.1 that the action of the BPS monopole is $-\langle z \rangle$ and the KK monopole action is $-2\pi i\tau + \langle z \rangle$, and also that in the quantum vacuum $\langle z \rangle = \pi i(\tau + \nu)$, so we can sum the contributions of both monopoles to find the gluino condensate,

$$\left\langle \frac{\mathrm{Tr}\,\lambda\lambda}{16\pi^2} \right\rangle = \frac{4\mu^3}{16\pi^2} \frac{4\pi^2}{g^2} e^{\pi i(\tau+\nu)} = \frac{\mu^3}{g^2} e^{\pi i(\tau+\nu)}. \tag{5.52}$$

This result is independent of R, so analytic continuation to all values including the limit $R \to \infty$ is trivial.

We can rewrite the expression (5.52) to make its renormalisation group invariance manifest. The solution of the exact Callan-Symanzik equation [22] in this pure $SU(2)$ gauge theory with $b_0 = 6$ is

$$\frac{\mu^6}{g^4(\mu)} \exp\left(-\frac{8\pi^2}{g^2(\mu)} + i\vartheta(\mu) \right) = \Lambda^6, \tag{5.53}$$

with Λ the dynamically generated scale in the Pauli-Villars regularisation scheme. Therefore,

$$\left\langle \frac{\mathrm{Tr}\,\lambda\lambda}{16\pi^2} \right\rangle = \Lambda^3 e^{i\pi\nu}, \tag{5.54}$$

which agrees with the WCI result for the gluino condensate.

Sending $\vartheta \mapsto \vartheta + 2\pi$ implies $\nu \mapsto \nu + 1$. This is also the way to relate the two physically equivalent vacua, so we see that they can be labelled according to the phase of the gluino condensate.

5.2.2 The functional method

Our knowledge of the superpotential allows us to find the gluino condensate by an alternative method that uses a formal identity but does not require a further semiclassical calculation. The original action, written in terms of the vector superfield W_α, is

$$S_{\mathrm{micro}}[W] = \int d^4x \,\mathrm{Im} \left(\int d^2\theta \, \frac{\tau}{4\pi} \mathrm{Tr}\,(W^\alpha W_\alpha) \right). \tag{5.55}$$

We can upgrade from τ to a chiral superfield T which includes the complexified coupling constant as the VEV of the scalar component,

$$T = A_\tau + \sqrt{2}\theta\psi_\tau + \theta\theta\mathcal{F}_\tau, \tag{5.56}$$

$$\langle A_\tau \rangle = \tau = \frac{4\pi i}{g^2} + \frac{\vartheta}{2\pi}. \tag{5.57}$$

The action and partition function then become functionals of T,

$$S_{\text{micro}}[T, W] = \int d^4x \,\text{Im}\left(\int d^2\theta \,\frac{T}{4\pi}\text{Tr}\,(W^\alpha W_\alpha)\right), \tag{5.58}$$

and

$$Z[T] = \int dW \,\exp\left(iS_{\text{micro}}[T, W]\right). \tag{5.59}$$

The lowest dimensional component of Tr $(W^\alpha W_\alpha)$ is $-$Tr $(\lambda^\alpha\lambda_\alpha)$, so the action contains

$$-\int d^4x \,\frac{\mathcal{F}_\tau}{8\pi}\text{Tr}\,(\lambda^\alpha\lambda_\alpha), \tag{5.60}$$

and Hermitian conjugate (with $\mathcal{F}_\tau^\dagger$). Therefore,

$$\frac{1}{Z[T]}\frac{\delta Z[T]}{\delta\mathcal{F}_\tau}\bigg|_{T=\tau} = -\int dW \,\frac{1}{8\pi}\text{Tr}\,(\lambda^\alpha\lambda_\alpha)\exp\left(iS_{\text{micro}}[W]\right) = -\left\langle\frac{\text{Tr}\,\lambda\lambda}{8\pi}\right\rangle. \tag{5.61}$$

Now, supposing that we were able to integrate out the vector superfield, we would be left with an effective action involving T,

$$Z[T] = \exp\left(i\int d^4x \,2\,\text{Re}\left(\int d^2\theta \,\mathcal{W}_{\text{eff}}(T)\right)\right). \tag{5.62}$$

(This is in the general form of equation (5.30), but with no D term because there is no connection between the chiral and anti-chiral sectors in equation (5.58).) We can expand this in the same manner as equation (5.35),

$$\int d^4x \int d^2\theta \,\mathcal{W}_{\text{eff}}(T) = \int d^4x \,\left(\frac{\partial\mathcal{W}_{\text{eff}}(A_\tau)}{\partial A_\tau}\mathcal{F}_\tau + \frac{1}{2}\frac{\partial^2\mathcal{W}_{\text{eff}}(A_\tau)}{\partial A_\tau^2}\psi_\tau\psi_\tau\right), \tag{5.63}$$

which leads to

$$\frac{1}{Z[T]}\frac{\delta Z[T]}{\delta\mathcal{F}_\tau}\bigg|_{T=\tau} = i\frac{\partial\mathcal{W}_{\text{eff}}(\tau)}{\partial\tau}, \tag{5.64}$$

so we have a useful identity that relates the gluino condensate to the derivative of the effective action,

$$\left\langle \frac{\mathrm{Tr}\,\lambda\lambda}{16\pi^2} \right\rangle = -\frac{i}{2\pi}\frac{\partial \mathcal{W}_{\text{eff}}(\tau)}{\partial \tau}. \tag{5.65}$$

However, it is not possible to do the integration over W_α directly. Instead, we may identify[7]

$$\mathcal{W}_{\text{eff}}(\tau) = \frac{\langle \mathcal{W} \rangle}{2\pi R}, \tag{5.66}$$

where $\mathcal{W}$ is the superpotential from section 5.1. Evaluating $\langle \mathcal{W} \rangle$ by substituting equation (5.47) into (5.45), we have

$$\frac{\langle \mathcal{W} \rangle}{2\pi R} = \frac{2\mu^3}{g^2} e^{\pi i(\tau+\nu)}. \tag{5.67}$$

Now we can draw all of these results together and find

$$\left\langle \frac{\mathrm{Tr}\,\lambda\lambda}{16\pi^2} \right\rangle = \frac{\mu^3}{g^2} e^{\pi i(\tau+\nu)} = \Lambda^3 e^{i\pi\nu}, \tag{5.68}$$

which agrees with the directly calculated value.

[7] The factor $2\pi R$ is present because $\mathcal{W}$ is only integrated over three spatial dimensions in the Wilsonian effective action.

Chapter 6

The generalisation to any gauge group, and the inclusion of matter

The one monopole calculation of the gluino condensate in $\mathcal{N} = 1$ supersymmetric pure Yang-Mills theory with gauge group $SU(2)$, presented in chapter 5, can be generalised in two directions, as we shall describe in this chapter. Firstly, we may find the gluino condensate with any gauge group, and then we can consider the addition of matter to the theory. Note that the generalisation to gauge group $SU(N)$, including the properties of monopoles on the cylinder, is alternatively presented in [1], using results motivated by string theory.

For the second issue, addressed in section 6.2, we will introduce the ADS superpotential that describes the low energy dynamics of theories including matter, and show that in all cases it can be directly evaluated using either a one instanton or one monopole calculation.

6.1 The gluino condensate in any gauge group

In this section we shall generalise the calculation of the gluino condensate, found in chapter 5 for gauge group $SU(2)$, to any gauge group. The only possibilities are compact Lie groups, whose algebras are known to be direct sums of $U(1)$ algebras and simple Lie algebras. The simple Lie algebras were classified by Cartan and Killing into types A,

B, C, D, E, F and G, as summarised in appendix B. We shall restrict our attention to the corresponding simple Lie groups, because there are no complications arising from trivial combinations of these groups.

The strategy of this calculation is exactly the same as for $SU(2)$, so we rely on the full explanations in chapter 5, and just give a summary with the relevant formulae here. We will discuss the interesting aspects of the results in more detail.

6.1.1 Classical low energy dynamics

Recall from section 4.3.3 that the VEV may be chosen to be in the Cartan subalgebra,

$$\langle v_0 \rangle = -V_i H^i. \tag{6.1}$$

Then, the classically massless degrees of freedom are the components of fields that correspond to Cartan subalgebra generators, and furthermore the parts of those fields that are independent of the periodic coordinate, x_0. We shall write, for example,

$$\lambda(x_m) = \lambda_i(x_\mu) H^i + \text{massive fields}. \tag{6.2}$$

Using the following normalisation, as outlined in appendix B,

$$\mathrm{Tr}\,(H^i H^j) = \delta^{ij}, \tag{6.3}$$

the action for the classically massless fields is

$$S_0 = \frac{2\pi R}{g^2} \int d^3x \left(\frac{1}{(2\pi R)^2} \partial_\mu \varphi_i \partial^\mu \varphi_i + \frac{1}{2} v_{\mu\nu i} v^{\mu\nu}{}_i + 2i \overline{\lambda}_i \overline{\sigma}^\mu \partial_\mu \lambda_i \right), \tag{6.4}$$

where φ is the Wilson loop as before, $\varphi = \int dx_0\, v_0$. Also, the part of a theta term that depends on these fields is

$$S_\vartheta = -\frac{i\vartheta}{8\pi^2} \int d^3x\, \epsilon^{\mu\nu\rho} v_{\nu\rho i} \partial_\mu \varphi_i. \tag{6.5}$$

We have the freedom to add n three dimensional topological terms, with n parameters, $\{\sigma_i\}$,

$$S_\sigma = \frac{i}{4\pi} \int d^3x\, \sigma_i \epsilon^{\mu\nu\rho} \partial_\mu v_{\nu\rho i}. \tag{6.6}$$

The quantisation of magnetic charge in general shows that the vector of magnetic charges in each of the unbroken $U(1)$ groups should lie in the coroot lattice [61]

$$\frac{1}{4\pi}\int d^3x\,\epsilon^{\mu\nu\rho}\partial_\mu v_{\nu\rho i}\in\Lambda_R^*, \tag{6.7}$$

which implies that σ_i must be valued in $\frac{\mathbb{R}^n}{2\pi\cdot\Lambda_W\rtimes W}$, because a shift by a weight vector ω contributes a power of the factor $e^{2\pi i}=1$ to the partition function. This space is slightly different, for non-simply laced groups, to the set of inequivalent values of φ_i, $\frac{\mathbb{R}^n}{2\pi\cdot\Lambda_W^*\rtimes W}$ (see section 4.3.3).

We can promote each σ_i to be an auxiliary field, ensuring the Bianchi identity everywhere, then integrate by parts to make the action quadratic in the field strength,

$$S=\int d^3x\,\left(aB_{\mu i}B^\mu{}_i+b_{\mu i}B^\mu{}_i+c\right), \tag{6.8}$$

with

$$a=\frac{2\pi R}{g^2}, \tag{6.9}$$

$$b_{\mu i}=-\frac{i}{2\pi}\partial_\mu\left(\sigma_i+\frac{\vartheta}{2\pi}\varphi_i\right), \tag{6.10}$$

$$c=\frac{1}{2\pi R}\left(\frac{1}{g^2}\partial_\mu\varphi_i\partial^\mu\varphi_i+\left(\frac{2\pi R}{g}\right)^2 2i\overline{\lambda}_i\overline{\sigma}^\mu\partial_\mu\lambda_i\right), \tag{6.11}$$

Now we can use Gaussian integration to eliminate $B_{\mu i}$ in favour of σ_i as a dynamical scalar field, in a dual theory with action

$$\begin{aligned}S_{\text{dual}}=&\frac{1}{2\pi R}\int d^3x\,\left(\frac{1}{g^2}\partial_\mu\varphi_i\partial^\mu\varphi_i+\left(\frac{2\pi R}{g}\right)^2 2i\overline{\lambda}_i\overline{\sigma}^\mu\partial_\mu\lambda_i\right)\\&+\frac{1}{2\pi R}\int d^3x\,\frac{g^2}{16\pi^2}\left(\partial_\mu\sigma_i+\frac{\vartheta}{2\pi}\partial_\mu\varphi_i\right)^2.\end{aligned} \tag{6.12}$$

We again switch our attention to a complex scalar fields and unconventionally normalised fermionic fields,

$$z_i=-i(\tau\varphi_i+\sigma_i),\quad \psi_i=\frac{2^{\frac{7}{2}}\pi^2R}{g^2}\lambda_i, \tag{6.13}$$

in terms of which the action is

$$S_{\text{dual}}=\frac{1}{8\pi^2R}\int d^3x\,\frac{1}{\operatorname{Im}\tau}\left(\partial_\mu z_i^\dagger\partial^\mu z_i+i\overline{\psi}_i\overline{\sigma}^\mu\partial_\mu\psi_i\right). \tag{6.14}$$

This is invariant under the supersymmetry transformations,

$$\delta z_i = \sqrt{2}\xi\psi_i, \quad \delta\psi_i = \sqrt{2}i\sigma^\mu\bar{\xi}\partial_\mu z_i, \tag{6.15}$$

so it is more convenient to work with the complex chiral superfields, $X_i = z_i + \sqrt{2}\theta\psi_i + \theta\theta\mathcal{F}_i$, which have the classical action

$$S_{\text{dual}} = \frac{1}{8\pi^2 R}\int d^3x\, d^2\theta\, d^2\bar{\theta}\,\frac{1}{\operatorname{Im}\tau}X_i^\dagger X_i. \tag{6.16}$$

6.1.2 Quantum low energy dynamics

The most general supersymmetric action of n complex chiral superfields is

$$S_{\text{susy}} = \int d^3x\left(\int d^2\theta\, d^2\bar{\theta}\,\mathcal{K}(X, X^\dagger) + \int d^2\theta\,\mathcal{W}(X) + \int d^2\bar{\theta}\,\mathcal{W}^\dagger(X^\dagger)\right). \tag{6.17}$$

The bosonic potential arises from the terms

$$\int d^3x\left(-\mathcal{F}_i G^{ij}\mathcal{F}_j^\dagger + \mathcal{F}_i\frac{\partial\mathcal{W}(z)}{\partial z_i} + \frac{\partial\mathcal{W}^\dagger(z^\dagger)}{\partial z_i^\dagger}\mathcal{F}_i^\dagger\right), \tag{6.18}$$

where $G^{ij} = \frac{\partial^2\mathcal{K}(z,z^\dagger)}{\partial z_i\partial z_j^\dagger}$, and we also define $G_{ij} = \|G^{ij}\|^{-1}$, the matrix inverse of G^{ij}. The Euler-Lagrange equation for any $\mathcal{F}_i$ gives

$$\mathcal{F}_i^\dagger = G_{ij}\frac{\partial\mathcal{W}(z)}{\partial z_j}, \tag{6.19}$$

and substituting these relations, and their complex conjugates, back into the action shows that the potential is

$$\int d^3x\, V = \int d^3x\,\frac{\partial\mathcal{W}^\dagger(z^\dagger)}{\partial z_i^\dagger}G_{ij}\frac{\partial\mathcal{W}(z)}{\partial z_j}, \tag{6.20}$$

which is zero if and only if the F-flatness conditions,

$$\frac{\partial\mathcal{W}(z)}{\partial z_i} = 0, \tag{6.21}$$

are fulfilled.

Expanding the F-term as

$$\int d^3x\int d^2\theta\,\mathcal{W}(X) = \int d^3x\left(\frac{\partial\mathcal{W}(z)}{\partial z_i}\mathcal{F}_i + \frac{1}{2}\frac{\partial^2\mathcal{W}(z)}{\partial z_i\partial z_j}\psi_i\psi_j\right), \tag{6.22}$$

we can see the fermionic effective mass term,

$$\int d^3x \, \frac{1}{2} \left\langle \frac{\partial^2 \mathcal{W}(z)}{\partial z_i \partial z_j} \right\rangle \psi_i \psi_j. \tag{6.23}$$

In order to evaluate the effective mass matrix, we need to consider the correlation function

$$\langle \lambda_{\alpha i}(x) \lambda_{\beta j}(0) \rangle \,, \tag{6.24}$$

which in the limit of large $|x|$ has the form

$$\lim_{|x|\to\infty} \langle \lambda_{\alpha i}(x) \lambda_{\beta j}(0) \rangle = \left(\frac{g^2}{4\pi R} \right)^2 m_\lambda^{ij} \int d^3x' \, \left(\mathcal{S}_{\mathrm{F}}(x - x') \mathcal{S}_{\mathrm{F}}(-x') \right)_{\alpha\beta} . \tag{6.25}$$

The definition of m_λ^{ij} is that it appears in the term $\int d^3x \, \frac{1}{2} m_\lambda^{ij} \lambda_i \lambda_j$ in the effective action.

When calculating the contribution from the monopole associated with the root[1] α, we need to know the large $|x|$ limit of the adjoint fermion zero modes of that monopole,

$$\lambda_{\beta i}^{\mathrm{LD}}(x) = 4\pi (\mathcal{S}_{\mathrm{F}}(x - X)\xi)_\beta \alpha^{*i}, \tag{6.26}$$

and the correct measure, taken from appendix C,

$$\int d\mu_{1-\mathrm{mono}}^{\alpha} = \frac{\mu^3 R}{g^2} \left(\frac{L}{|\alpha|^2} \right) e^{-S} \int d^3X \int_0^{2\pi} d\Omega \int d^2\xi. \tag{6.27}$$

Combining these elements, we can find the effect of the monopole associated with the root α on the large $|x|$ limit of the correlation function,

$$\begin{aligned} \langle \lambda_{\alpha i}(x) \lambda_{\beta j}(0) \rangle_{[\alpha]} &\overset{|x|\to\infty}{\longrightarrow} \frac{\mu^3 R}{g^2} \left(\frac{L}{|\alpha|^2} \right) e^{-S} \int d^3X \int d\Omega \int d^2\xi \, \lambda_{\alpha i}^{\mathrm{LD}}(x) \lambda_{\beta j}^{\mathrm{LD}}(0) && (6.28) \\ &= \frac{2^4 \pi^3 \mu^3 R}{g^2} \left(\frac{L}{|\alpha|^2} \right) e^{-S} \alpha^{*i} \alpha^{*j} \int d^3X \, (\mathcal{S}_{\mathrm{F}}(x - X) \mathcal{S}_{\mathrm{F}}(-X))_{\alpha\beta} . && (6.29) \end{aligned}$$

We may read off the contribution to $\left\langle \frac{\partial^2 \mathcal{W}(z)}{\partial z_i \partial z_j} \right\rangle$, from

$$\left[\frac{2^4 \pi^3 \mu^3 R}{g^2} \left(\frac{4\pi R}{g^2} \right)^2 \left(\frac{L}{|\alpha|^2} \right) e^{-S} \right] \frac{(\alpha^* \cdot \lambda)^2}{2} = \left[\frac{2\pi \mu^3 R}{g^2} \left(\frac{L}{|\alpha|^2} \right) e^{-S} \right] \frac{(\alpha^* \cdot \psi)^2}{2} . \tag{6.30}$$

[1]Recall from section 4.3.3 that the BPS monopoles correspond to the simple roots, the KK or affine monopole to the lowest root, $\alpha_{(0)} = -\theta$.

The action of the BPS monopole associated with the simple root α_i is $S[\alpha_{(i)}] = -\alpha^*_{(i)} \cdot \langle z \rangle$, and the action of the KK or affine monopole is $S[\alpha_{(0)}] = -\alpha^*_{(0)} \cdot \langle z \rangle - 2\pi i \tau$, as given in section 4.3.3, so we can sum all $n+1$ monopole contributions into

$$\left\langle \frac{\partial^2 \mathcal{W}(z)}{\partial z_j \partial z_k} \right\rangle = \frac{2\pi\mu^3 R}{g^2} \sum_{i=0}^{n} \left(\frac{L}{|\alpha_{(i)}|^2} \right) \alpha^{*j}_{(i)} \alpha^{*k}_{(i)} \exp\left(\alpha^*_{(i)} \cdot \langle z \rangle + 2\pi i \tau \delta_{i0} \right), \tag{6.31}$$

and use this to reconstruct the full superpotential,

$$\mathcal{W}(X) = \frac{2\pi\mu^3 R}{g^2} \sum_{i=0}^{n} \left(\frac{L}{|\alpha_{(i)}|^2} \right) \exp\left(\alpha^*_{(i)} \cdot X + 2\pi i \tau \delta_{i0} \right). \tag{6.32}$$

The form of this superpotential is that of a twisted affine Toda potential, as was predicted for $\mathcal{N} = 1$ gauge theories on $\mathbb{R}^3 \times S^1$ by Katz and Vafa in [66], on the basis of string theory calculations (see also [67]). However, this result does not include the field theory interpretation of the variables of the superpotential, which is supplied by the expression above. The following linear shift in the chiral superfield,

$$X \mapsto X - \sum_{i=1}^{n} \log\left(\frac{L}{|\alpha_{(i)}|^2} \right) \omega^{(i)} + \frac{1}{c_2} \left(2\pi i \tau + \sum_{i=1}^{n} m^i \log\left(\frac{L}{|\alpha_{(i)}|^2} \right) \right) \rho, \tag{6.33}$$

renders the superpotential into the standard form for a twisted affine Toda potential, as written by Katz and Vafa. It is based on the twisted affine algebra, which coincides with the affine algebra for the simply laced algebras, and can be found for the non-simply laced algebras either by changing the roots into coroots [68], or by the imposition of an outer automorphism on a simply laced algebra before affinization [69]. In the first method, the twisted affine algebra of X_n is denoted $X_n^{(1)*}$, as the dual of the untwisted version $X_n^{(1)}$. In the latter case, the outer automorphism corresponds to a symmetry of the Dynkin diagram of the simply laced algebra, Y_n, with order 2 except for D_4 where it is 3. Then, following the notation of Kac [70], the resultant twisted affine algebra is written $Y_n^{(2)}$ or $Y_n^{(3)}$ respectively, as summarised in table 6.1.

A related superpotential on $\mathbb{R}^3 \times S^1$ was calculated by Dorey [71], in an $\mathcal{N} = 1$ supersymmetric theory with a specific matter content, obtained by adding masses to $\mathcal{N} = 4$ supersymmetric pure gauge theory in order to break all but one supersymmetry. This superpotential is also generated by monopoles, and takes the form of a Calogero-Moser potential. Renormalisation group decoupling to make the matter fields irrelevant corresponds exactly to the well known limit of the Calogero-Moser potential that produces the affine Toda potential [68].

Simple Lie algebra	Twisted affine algebra
B_n	$B_n^{(1)*} = A_{2n-1}^{(2)}$
C_n	$C_n^{(1)*} = D_{n+1}^{(2)}$
F_4	$F_4^{(1)*} = E_6^{(2)}$
G_2	$G_2^{(1)*} = D_4^{(3)}$

Table 6.1: The twisted affine algebras.

6.1.3 Calculation of the gluino condensate

Before finding the gluino condensate, we must first determine the true quantum vacuum by solving the F-flatness condition,

$$\frac{\partial \mathcal{W}(z)}{\partial z} = 0. \tag{6.34}$$

It is most convenient to work with the n variables $\zeta_i = \alpha^*_{(i)} \cdot z$, which are independent because the simple coroots $\{\alpha^*_{(i)}\}$ form a basis. Using $\alpha^*_{(0)} = -\theta = -\sum_{i=1}^n m^i \alpha^*_{(i)}$, and $|\alpha_{(0)}|^2 = |\theta|^2 = L$, the superpotential can be written

$$\mathcal{W}(z) = \frac{2\pi\mu^3 R}{g^2}\left(\exp\left(2\pi i\tau - \sum_{i=1}^n m^i\zeta_i\right) + \sum_{i=1}^n \left(\frac{L}{|\alpha_{(i)}|^2}\right)\exp(\zeta_i)\right). \tag{6.35}$$

Differentiating with respect to ζ_i gives

$$\left(\frac{L}{|\alpha_{(i)}|^2}\right)\exp\langle\zeta_i\rangle = m^i \exp\left(2\pi i\tau - \sum_{i=1}^n m^i\langle\zeta_i\rangle\right) \equiv m^i\Xi, \tag{6.36}$$

so we can immediately see that, in the quantum vacuum,

$$\langle\mathcal{W}(z)\rangle = \frac{2\pi\mu^3 R}{g^2}\left(1 + \sum_{i=1}^n m^i\right)\Xi = \frac{2\pi\mu^3 R}{g^2}c_2\Xi. \tag{6.37}$$

The exponential factor Ξ may be found by repeatedly applying equation (6.36), so

$$\Xi = \exp\left(2\pi i\tau - \sum_{i=1}^n m^i\langle\zeta_i\rangle\right) \tag{6.38}$$

$$= \exp(2\pi i\tau)\prod_{i=1}^n \left(\exp\langle\zeta_i\rangle\right)^{-m^i} \tag{6.39}$$

$$= \exp(2\pi i\tau)\left[\prod_{i=1}^n \left(\frac{m^i|\alpha_{(i)}|^2}{L}\right)^{-m^i}\right]\Xi^{-\sum_{i=1}^n m^i}. \tag{6.40}$$

Therefore,

$$\Xi^{c_2} = \exp(2\pi i\tau) \prod_{i=1}^{n} \left(\frac{m^i |\alpha_{(i)}|^2}{L} \right)^{-m^i}, \tag{6.41}$$

$$\Xi = \exp\left(\frac{2\pi i(\tau+\nu)}{c_2} \right) \prod_{i=1}^{n} \left(\frac{m^i |\alpha_{(i)}|^2}{L} \right)^{-\frac{m^i}{c_2}}. \tag{6.42}$$

with $\nu \in \mathbb{Z}$. This integer labels the c_2 vacua, related to each other by $\vartheta \mapsto \vartheta + 2\pi$, and predicted by Witten's index [40].

The functional method of section 5.2.2 is group independent and therefore trivial to generalise, so we may immediately find the gluino condensate in any gauge group,

$$\left\langle \frac{\operatorname{Tr}\lambda\lambda}{16\pi^2} \right\rangle = -\frac{i}{4\pi^2 R} \frac{\partial \langle \mathcal{W} \rangle}{\partial \tau} = \frac{\mu^3}{g^2} \exp\left(\frac{2\pi i(\tau+\nu)}{c_2} \right) \prod_{i=1}^{n} \left(\frac{m^i |\alpha_{(i)}|^2}{L} \right)^{-\frac{m^i}{c_2}}. \tag{6.43}$$

This is independent of R, and so may be automatically continued and taken to be the result for any value, in particular the $R \to \infty$ limit. Using the solution to the exact Callan-Symanzik equation,

$$\frac{\mu^{b_0}}{g^{\frac{2b_0}{3}}(\mu)} \exp\left(-\frac{8\pi^2}{g^2(\mu)} + i\vartheta(\mu) \right) = \Lambda^{b_0}, \tag{6.44}$$

where $b_0 = 3c_2$, for any pure gauge theory, we can write the gluino condensate in terms of the dynamically generated scale, Λ, in the Pauli-Villars regularisation scheme,

$$\left\langle \frac{\operatorname{Tr}\lambda\lambda}{16\pi^2} \right\rangle = \Lambda^3 e^{\frac{2\pi i\nu}{c_2}} \prod_{i=1}^{n} \left(\frac{m^i |\alpha_{(i)}|^2}{L} \right)^{-\frac{m^i}{c_2}}. \tag{6.45}$$

The Lie algebra data given in tables B.1 and B.2 is sufficient to evaluate this formula for all the simple Lie groups, and the results are shown in table 6.2 (ignoring the phase factor that distinguishes the vacua). Note that although the algebras of $SO(2n)$ and $SO(2n+1)$ are very different, the expression for the gluino condensate in $SO(N)$ is a single formula in terms of N, irrespective of whether N is even or odd.

The values for the classical groups are in full agreement with those calculated by Finnell and Pouliot [37] in a WCI approach, and such consistency over a range of non-trivial numbers, from independent methods, is good evidence that WCI methods are sound.

Gauge group	$\Lambda^{-3}\left\langle\frac{\mathrm{Tr}\,\lambda\lambda}{16\pi^2}\right\rangle$
$SU(N)$	1
$SO(N)$	$2^{-1+\frac{4}{N-2}}$
$USp(N)$	$2^{1-\frac{2}{N+1}}$
G_2	$2^{-\frac{1}{2}}3^{\frac{1}{4}}$
F_4	$2^{-\frac{1}{9}}3^{-\frac{1}{3}}$
E_6	$2^{-\frac{1}{2}}3^{-\frac{1}{4}}$
E_7	$2^{-\frac{7}{9}}3^{-\frac{1}{3}}$
E_8	$2^{-\frac{13}{15}}3^{-\frac{2}{5}}5^{-\frac{1}{6}}$

Table 6.2: The gluino condensate for all simple Lie groups.

The results for the exceptional groups are new predictions; the monopole method is at present the only known way to determine these numbers.

In the direct semiclassical evaluation of the gluino condensate, using the correct quantum vacuum determined by the F-flatness condition, the monopole associated with the root α contributes

$$\langle\mathrm{Tr}\,\lambda\lambda\rangle_{[\alpha]} = 2\mu^3\left(\frac{L}{|\alpha|^2}\right)\mathrm{Re}(S[\alpha])\,e^{-S[\alpha]}. \tag{6.46}$$

Using $e^{-S[\alpha_{(0)}]} = \Xi$, and $e^{-S[\alpha_{(i)}]} = \left(\frac{m^i|\alpha_{(i)}|^2}{L}\right)\Xi$, we can sum over the contributions of all of the monopoles and find

$$\left\langle\frac{\mathrm{Tr}\,\lambda\lambda}{16\pi^2}\right\rangle = \frac{\mu^3}{g^2}\Xi, \tag{6.47}$$

which is exactly the same as the result found by the functional method.

6.2 The ADS superpotential

Another highly important result in the context of $\mathcal{N}=1$ supersymmetric gauge theories is the superpotential found by Affleck, Dine and Seiberg [72]. They considered an $SU(N)$ gauge theory with N_f flavours of matter, where $N_f < N$. Each flavour consists of a chiral superfield Q that transforms according to the fundamental representation of the gauge group, and a chiral superfield $\tilde{Q}$ that transforms in the conjugate fundamental. This prescription is used so that the fermion from Q, ψ_L, and the fermion from $\tilde{Q}^\dagger$, $\overline{\psi}_R$, combine to form a single Dirac fermion, such as would be implied by the term flavour in non-supersymmetric gauge theories.

The classical global symmetry group of the theory is

$$SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A \times U(1)_R, \tag{6.48}$$

which is equivalent to the non-supersymmetric version, except for the addition of the $U(1)_R$ factor. This is related to the $U(1)$ group of R-symmetry transformations, which are induced by changing the phase of the supersymmetry generators. Just as in a non-supersymmetric theory, the quantum global symmetry group does not contain the $U(1)_A$; this symmetry has an anomaly, produced by the semiclassical configurations of the theory. R-symmetry is similarly anamolous, and we choose $U(1)_R$ to be the anomaly-free combination of the $U(1)_A$ and R-symmetry.

The superpartners of the matter fermions are $2NN_f$ complex scalars, which may develop VEVs that generically break the gauge group to $SU(N-N_f)$. In addition, they break the global symmetry group to

$$SU(N_f)_D \times U(1)_V, \tag{6.49}$$

where $SU(N_f)_D$ is the diagonal subgroup of $SU(N_f)_L \times SU(N_f)_R$. Therefore, $N_f(2N-N_f)$ of the real scalar degrees of freedom are absorbed under the Higgs mechanism, to allow for the right number of massive gauge bosons, and N_f^2+1 further real scalars will be Goldstone bosons that will be classically massless, associated with massless fermionic Goldstinos by supersymmetry. This last set must be gauge invariant fields, and indeed they are contained in the chiral superfields $\tilde{Q}^{fi}Q_{ig}$ (i is a gauge index, f and g label flavours). Note, however, that one set of these Goldstone fields, the one associated with the anomalous $U(1)_A$ symmetry, will gain a semiclassically generated mass, while the others must remain massless.

Affleck, Dine and Seiberg determined the low energy effective action of these classically massless fields, encoded in the superpotential that is obtained after all massive and high virtuality fields, and also the massless gauge fields, have been integrated out. This is known as the ADS superpotential, and like the superpotential we considered here for supersymmetric pure gauge theory on $\mathbb{R}^3 \times S^1$, it is not calculated by direct integration. Instead, requiring invariance under the remaining symmetries shows that it must be of the form

$$\mathcal{W}^{\mathrm{ADS}}_{N,N_f} = C_{N,N_f} \left(\frac{\Lambda^{3N-N_f}_{N,N_f}}{\det \tilde{Q}^f Q_g} \right)^{\frac{1}{N-N_f}}, \tag{6.50}$$

where Λ_{N,N_f} is the dynamically generated scale, and $3N - N_f = b_0$ is the first coefficient of the beta function, in the theory with gauge group $SU(N)$ and N_f flavours. The dimensionless constant C_{N,N_f} is to be determined. The strategy is to isolate a part that is accessible to direct evaluation, and use this to reconstruct the full superpotential. Using instantons in $\mathbb{R}^4$, this may only be achieved in the case $N_f = N - 1$, where the gauge group is completely broken and the coupling is controllable. This is sufficient, however, because starting from that result, renormalisation group decoupling can be applied to flow to theories with lower numbers of flavours, and determine the ADS superpotential for all $N_f < N$. The part of the superpotential that is calculated is in fact the semiclassically generated mass of the $U(1)_A$ Goldstino, using the asymptotic behaviour of a two point function, as in the method employed in section 5.1.4. The only contribution is from one instanton configurations, and it is found that $C_{N,N_f} = N - N_f$.

For $N_f < N - 1$, calculations using $\mathbb{R}^4$ instantons are not reliable, because there is an unbroken non-abelian subgroup of the gauge group, which leads to strong coupling effects. However, if we modify the space to $\mathbb{R}^3 \times S^1$, we find that monopoles can be used to generate the ADS superpotential, as we shall now discuss.

Suppose that the massive and high virtuality fields have been integrated out, leaving the $SU(N - N_f)$ gauge fields, and the chiral superfields $\tilde{Q}^f Q_g$. It is convenient to change variables from the latter to the following matrix of superfields[2],

$$\Phi = \frac{v}{\sqrt{2}} \log \frac{\tilde{Q}Q}{v^2}. \tag{6.51}$$

The effective action at this level may be expanded in inverse powers of the VEV parameter of the scalars, v, and, as observed in [72], there is only one dimension five operator that is invariant under the unbroken gauge and global symmetries (its form is

[2] Affleck, Dine and Seiberg used a linearised version of the superfield matrix Φ.

completely specified by these), so we can say

$$S_{\text{eff}} = \int d^4x \, \text{Im} \int d^2\theta \left(\frac{\tau}{4\pi} \text{Tr}\,(W^\alpha W_\alpha) + \frac{\sqrt{2}i}{8\pi^2 v} \Phi^f{}_f \text{Tr}\,(W^\alpha W_\alpha) + \mathcal{O}\left(\frac{1}{v^2}\right)\right). \quad (6.52)$$

Here S_{eff} is the effective action of the classically massless fields, which include W_α, the $SU(N-N_f)$ field strength superfield. Integrating this out is equivalent to finding the effective action for the auxiliary chiral superfield T in a theory like that given in equation (5.58), but where T has the value

$$T = \tau + \frac{\sqrt{2}i}{2\pi v} \Phi^f{}_f. \quad (6.53)$$

We know, from comparing equation (6.37), with $c_2 = N - N_f$ for the unbroken gauge group $SU(N-N_f)$, to equation (5.66), that monopoles in $SU(N-N_f)$ will generate

$$\mathcal{W}_{\text{eff}} = (N - N_f) \frac{\mu^3}{g^2} \exp\left(\frac{2\pi i T}{N - N_f}\right) \quad (6.54)$$

$$= (N - N_f) \frac{\mu^3}{g^2} \exp\left(\frac{2\pi i \tau}{N - N_f}\right) \left(\frac{v^{2N_f}}{\det \tilde{Q}Q}\right)^{\frac{1}{N-N_f}} \quad (6.55)$$

$$= (N - N_f) \Lambda^3_{N-N_f,0} \left(\frac{v^{2N_f}}{\det \tilde{Q}Q}\right)^{\frac{1}{N-N_f}}, \quad (6.56)$$

and using the renormalisation group decoupling equation $\Lambda_{N,N_f}^{3N-N_f} = v^{2N_f} \Lambda_{N-N_f,0}^{3(N-N_f)}$, we find

$$\mathcal{W}_{\text{eff}} = (N - N_f) \left(\frac{\Lambda_{N,N_f}^{3N-N_f}}{\det \tilde{Q}Q}\right)^{\frac{1}{N-N_f}} = \mathcal{W}^{\text{ADS}}_{N,N_f}. \quad (6.57)$$

This shows that the ADS superpotential is closely linked to gluino condensation, and indeed its derivation in [72] is also a WCI calculation of the gluino condensate with gauge group $SU(N)$. Analogous results have also been determined for the other classical groups [43], and these are all part of an extended family of superpotentials, in $\mathcal{N} = 1$ supersymmetric theories with different gauge groups and various types of matter. They can be connected in many ways through renormalisation group flows, and are all mutually consistent. See the reviews [37] and [73] for discussions and an introduction to the literature.

Chapter 7

Two monopole calculations

7.1 Introduction

We have introduced a strategy that in principle allows particular correlation functions in supersymmetric theories to be calculated exactly; modify space from $\mathbb{R}^4$ to the cylinder $\mathbb{R}^3 \times S^1$, and choose the radius of the circle to be small so that the coupling is also small. Then any correlation function is equal to a sum of contributions, from conventional perturbation theory and semiclassical configurations. For correlation functions of F-terms, there is no conventional perturbation theory part, and the one loop semiclassical approximation gives the full result. The connection between the coupling and the radius means that the answer can then be analytically continued back to the case of a large radius. Furthermore, the relevant semiclassical configurations that may contribute are known; for gauge group $SU(2)$ they are all combinations of the two types of (multi-)monopoles on the cylinder[1]. In chapter 5 we employed this formalism to calculate the gluino condensate in $SU(2)$, and found agreement with the WCI value. On the other hand, the SCI value of the gluino condensate, calculated via the one instanton contribution to

$$\left\langle \frac{\operatorname{Tr}\lambda\lambda(x)}{16\pi^2}\frac{\operatorname{Tr}\lambda\lambda(0)}{16\pi^2}\right\rangle, \tag{7.1}$$

is believed to be incorrect because the contributions of some other configurations are neglected, and then cluster decomposition is applied to a part of the correlation function

[1] In general there are $n+1$ types of monopoles where n is the rank of the gauge group.

rather than the full result.

On the cylinder, the analogue of the one instanton is the caloron, which is a combination of a BPS monopole and a KK monopole. However, there are two more configurations on $\mathbb{R}^3 \times S^1$ that have four adjoint fermion zero modes, and so are relevant for calculating this correlation function. They are combinations of two of the same type of monopole, either BPS or KK. These two monopole configurations do not have useful limiting forms in $\mathbb{R}^4$; they would both have infinite action for example, just like the one monopoles. Similarly to the way that one monopoles aid our understanding of the contributions of the badly behaved $\mathbb{R}^4$ instanton partons, though, the two monopole solutions help us to understand what the missed configurations are in the SCI approach. Ideally, we want a quantitative explanation, so we should attempt to calculate the two monopole contributions to the correlation function above, and show that they make up the deficit. This chapter describes work undertaken in preparation for such a calculation. Unfortunately, however, although some progress was made, two monopole calculations eventually proved to be technically impossible, which brought the monopole programme to a disappointingly early end.

Most of this chapter is concerned with an investigation of various aspects of the Nahm construction of multi-monopole solutions [74]. The one monopole fields are simple enough to be found by conventional analytic means and expressed in terms of elementary functions, but for higher winding numbers such a direct approach is unproductive. All multi-monopole solutions can, at least in principle, be constructed by a method given by Nahm. In practice, only the form of the two monopole fields have been worked out, they are much more complicated than the one monopole expressions, and they involve Jacobi elliptic functions.

This situation is analogous to that of the present knowledge of multi-instanton solutions, which is perhaps not surprising as the Nahm construction is a generalisation of the instanton construction by Atiyah, Drinfeld, Hitchin and Manin [14]. Both the Nahm and ADHM constructions originated from twistor theory, where the self-duality condition obeyed by instantons and monopoles is reformulated as a powerful statement of vanishing curvature.

In the case of the ADHM construction, the problem of finding finite action self-dual Yang-Mills solutions (or solving a set of coupled, non-linear, first-order partial differential equations) is reduced to one of solving non-linear algebraic equations. This is achieved by expressing the gauge field and field strength in terms of vectors and matrices with dimensions related to the instanton number. The ADHM equations have only been solved for $k \leqslant 3$, but all multi-instanton solutions are implicitly determined by the ADHM construction.

In a similar construction giving finite energy monopole solutions in $\mathbb{R}^4$, the infinite action of the monopoles suggests that linear algebra with an infinite dimensional vector space must be used. Nahm generalised the ADHM construction in exactly this way, and as we shall see, the equivalent of the algebraic equations of ADHM is a set of non-linear ordinary differential equations collectively known as the Nahm equation.

The ADHM construction has been described in [16, 75, 28, 29], using language suited to theoretical physicists. We attempt to do the same for the Nahm construction here, so we shall not prove that every monopole can be built in this way (this was done in [76]), instead we will use a strong assumption to demonstrate how the construction works. First, we give a brief summary of the ADHM construction, because analogy with it motivates our approach to the Nahm construction, but for more detailed explanations we recommend the references above.

Throughout this chapter we will consider only gauge group $SU(2)$. Following the tradition of the original work on the ADHM and Nahm constructions, we shall work with an anti-Hermitian gauge field. A Hermitian gauge field may be recovered through $v_m^{\mathrm{Herm}} = iv_m^{\mathrm{anti-Herm}}$.

7.2 The ADHM construction

7.2.1 Overview

The gauge field of a multi-instanton is written in the ADHM formalism as a generalisation of the pure gauge form of a trivial ($k = 0$) solution,

$$v_m{}^{\dot{\alpha}}{}_{\dot{\gamma}} = \overline{U}^{\lambda\dot{\alpha}\beta}\partial_m U_{\lambda\beta\dot{\gamma}}, \tag{7.2}$$

where λ is implicitly summed over $0, 1, \ldots, k$ for a k-instanton solution, and

$$\overline{U}^{\lambda\dot{\alpha}\beta} U_{\lambda\beta\dot{\gamma}} = \delta^{\dot{\alpha}}{}_{\dot{\gamma}}. \tag{7.3}$$

In the following, we shall often drop the two-valued indices for brevity. The field strength is

$$v_{mn} = \partial_{[m} v_{n]} + v_{[m} v_{n]}, \tag{7.4}$$

where for now $A_{[mn]} = A_{mn} - A_{nm}$. Substituting the gauge field and making use of the orthonormality condition above, we have

$$v_{mn} = \partial_{[m}\left(\overline{U}^{\lambda}\partial_{n]}U_{\lambda}\right) + \left(\overline{U}^{\lambda}\partial_{[m}U_{\lambda}\right)\left(\overline{U}^{\kappa}\partial_{n]}U_{\kappa}\right) \tag{7.5}$$

$$= \left(\partial_{[m}\overline{U}^{\lambda}\right)\left(\partial_{n]}U_{\lambda}\right) - \left(\partial_{[m}\overline{U}^{\lambda}\right)U_{\lambda}\overline{U}^{\kappa}\left(\partial_{n]}U_{\kappa}\right) \tag{7.6}$$

$$= \left(\partial_{[m}\overline{U}^{\lambda}\right)\left[\delta_{\lambda}{}^{\kappa} - U_{\lambda}\overline{U}^{\kappa}\right]\left(\partial_{n]}U_{\kappa}\right). \tag{7.7}$$

Note that $U_{\lambda}\overline{U}^{\kappa}$ and $\delta_{\lambda}{}^{\kappa} - U_{\lambda}\overline{U}^{\kappa}$ are orthogonal projection operators, which obey

$$\left(U_{\lambda}\overline{U}^{\kappa}\right)\left(U_{\kappa}\overline{U}^{\iota}\right) = U_{\lambda}\overline{U}^{\iota}, \tag{7.8}$$

$$U_{\iota}\overline{U}^{\lambda}\left[\delta_{\lambda}{}^{\kappa} - U_{\lambda}\overline{U}^{\kappa}\right] = 0, \quad \left[\delta_{\lambda}{}^{\kappa} - U_{\lambda}\overline{U}^{\kappa}\right]U_{\kappa}\overline{U}^{\iota} = 0. \tag{7.9}$$

These conditions are preserved if we identify

$$\left[\delta_{\lambda}{}^{\kappa} - U_{\lambda}\overline{U}^{\kappa}\right] = \Delta_{\lambda}{}^{l} F_{l}{}^{j} \overline{\Delta}_{j}{}^{\kappa}, \tag{7.10}$$

where $\overline{\Delta}$ annihilates U,

$$\overline{\Delta}_{j}{}^{\kappa} U_{\kappa} = 0, \quad \overline{U}^{\lambda}\Delta_{\lambda}{}^{l} = 0, \tag{7.11}$$

and F is the inverse of $\overline{\Delta}\Delta$ (which must exist as $\overline{\Delta}\Delta$ is Hermitian and positive definite),

$$\overline{\Delta}_{j}{}^{\kappa}\Delta_{\kappa}{}^{l} = (F^{-1})_{j}{}^{l}. \tag{7.12}$$

There should be k independent vectors orthogonal to the ADHM vector U, so the Latin indices j, l run over $1, \ldots, k$. The full index structures of the operators Δ and $\overline{\Delta}$ are

$$\Delta_{\lambda}{}^{l}{}_{\alpha\dot{\beta}} \quad \text{and} \quad \overline{\Delta}_{j}{}^{\kappa\dot{\alpha}\beta}, \tag{7.13}$$

while the matrix F has indices $F_l{}^j{}_{\dot\alpha}{}^{\dot\beta}$.

We assume the following properties of Δ, $\overline{\Delta}$ and F,

$$\partial_m\Delta \propto \sigma_m, \quad \partial_m\overline{\Delta} \propto \overline{\sigma}_m, \quad [F,\sigma_m] = 0. \tag{7.14}$$

The first two imply that Δ and $\overline{\Delta}$ depend linearly on the space point x,

$$\Delta_\lambda{}^l = a_\lambda{}^l + b_\lambda{}^l x^m \sigma_m, \tag{7.15}$$

$$\overline{\Delta}_j{}^\kappa = \overline{a}_j{}^\kappa + x^m \overline{\sigma}_m \overline{b}_j{}^\kappa. \tag{7.16}$$

The matrix a follows the index structure of Δ, while b is $b_\lambda{}^l{}_\alpha{}^\beta$ in full. Both are obviously independent of x. To ensure the third property we require that F factorises as

$$F_l{}^j{}_{\dot\alpha}{}^{\dot\beta} = f_l{}^j \delta_{\dot\alpha}{}^{\dot\beta}. \tag{7.17}$$

The features of linearity and factorisation are crucial to the ADHM construction, and also to the Nahm construction.

Using the relations above, we can develop the field strength into

$$v_{mn} = \left(\partial_{[m}\overline{U}^\lambda\right)\Delta_\lambda{}^l f_l{}^j \overline{\Delta}_j{}^\kappa \left(\partial_{n]}U_\kappa\right) \tag{7.18}$$

$$= \overline{U}^\lambda\left(\partial_{[m}\Delta_\lambda{}^l\right) f_l{}^j \left(\partial_{n]}\overline{\Delta}_j{}^\kappa\right) U_\kappa \tag{7.19}$$

$$= \overline{U}^\lambda b_\lambda{}^l \sigma_{[m} f_l{}^j \overline{\sigma}_{n]} \overline{b}_j{}^\kappa U_\kappa \tag{7.20}$$

$$= 4\overline{U}^\lambda b_\lambda{}^l f_l{}^j \sigma_{mn} \overline{b}_j{}^\kappa U_\kappa, \tag{7.21}$$

which is self-dual by virtue of the self-duality of σ_{mn}.

The necessity that F, and therefore also F^{-1}, is proportional to the 2×2 identity matrix leads to the following conditions on the matrices a and b,

$$\overline{a}^{\dot\alpha} a_{\dot\beta} \propto \delta^{\dot\alpha}{}_{\dot\beta}, \tag{7.22}$$

$$\overline{b}_\alpha b^\beta \propto \delta_\alpha{}^\beta, \tag{7.23}$$

$$\overline{a}^{\dot\alpha} b^\beta = \overline{b}^\beta a^{\dot\alpha}. \tag{7.24}$$

These are the ADHM equations. They can be simplified by using the symmetries of the ADHM construction to bring b into canonical form,

$$b_\lambda{}^l{}_\alpha{}^\beta = \delta_\lambda{}^l \delta_\alpha{}^\beta, \tag{7.25}$$

where the Kronecker delta is understood to be zero when $\lambda = 0$. The transformations under which the ADHM formalism is invariant are

$$U \mapsto \Lambda U, \quad \overline{U} \mapsto \overline{U}\Lambda^\dagger, \tag{7.26}$$

$$\Delta \mapsto \Lambda\Delta, \quad \overline{\Delta} \mapsto \overline{\Delta}\Lambda^\dagger, \tag{7.27}$$

with $\Lambda \in U(2k+2)$ (unitary to preserve the orthonormality of U), and

$$\Delta \mapsto \Delta B^\dagger, \quad \overline{\Delta} \mapsto B\overline{\Delta}, \quad f \mapsto BfB^\dagger, \tag{7.28}$$

where $B \in U(k)$ (k-dimensional not $2k$-dimensional to maintain the factorisation of F, and unitary to keep f Hermitian, as it must be because $\overline{\Delta}\Delta$ is Hermitian). Gauge transformations $v_m \mapsto \Omega v_m \Omega^\dagger + \Omega\partial_m\Omega^\dagger$ act just on U as $U \mapsto U\Omega^\dagger$.

7.2.2 Adjoint fermion zero modes in ADHM

The ADHM construction also provides us with the form of the zero modes of any multi-instanton. This includes, of obvious interest in this thesis, the adjoint fermion zero modes that solve

$$\overline{\sigma}^{m\dot{\alpha}\alpha} D_m \lambda_\alpha = 0. \tag{7.29}$$

The covariant derivative acts in the appropriate way given that λ_α transforms under the adjoint representation,

$$D_m\lambda_\alpha = \partial_m\lambda_\alpha + [v_m, \lambda_\alpha]\,. \tag{7.30}$$

If we use the ADHM expression for the gauge field then we find

$$D_m\lambda_\alpha = \partial_m\lambda_\alpha + \left[\overline{U}\partial_m U, \lambda_\alpha\right] \tag{7.31}$$

$$= \overline{U}\left(\partial_m\left[U\lambda_\alpha\overline{U}\right]\right)U. \tag{7.32}$$

Note that we are now suppressing *all* non-essential indices.

In the ADHM formalism, the adjoint fermion zero modes are written in terms of the ADHM data we have already encountered, plus a constant Grassmannian matrix ${\mathcal{M}_{\beta\lambda}}^{l}$,

$$(\lambda_\alpha)^{\dot{\beta}}{}_{\dot{\gamma}} = \overline{U}^{\dot{\beta}}\mathcal{M}f\bar{b}_\alpha U_{\dot{\gamma}} - \overline{U}^{\dot{\beta}}b_\alpha f\overline{\mathcal{M}}U_{\dot{\gamma}}. \tag{7.33}$$

In order to proceed with the calculation of $D_m\lambda_\alpha$, we apply the following useful relations. First,

$$\overline{U}\partial_m\left(U\overline{U}\right) = -\overline{U}\partial_m\left(\Delta f\overline{\Delta}\right) \tag{7.34}$$

$$= -\overline{U}\left(\partial_m\Delta\right)f\overline{\Delta} \tag{7.35}$$

$$= -\overline{U}b\sigma_m f\overline{\Delta}, \tag{7.36}$$

and its conjugate,

$$\partial_m\left(U\overline{U}\right)U = -\Delta f\overline{\sigma}_m\overline{b}U, \tag{7.37}$$

then the orthonormality condition $\overline{U}^{\dot\alpha}U_{\dot\beta} = \delta^{\dot\alpha}{}_{\dot\beta}$. Finally, we need to know the derivative of the matrix f, but rather than attempt to differentiate it directly, we rearrange the identity $\partial_m\left(f^{-1}f\right) = 0$, to find

$$\partial_m f = -f\left(\partial_m f^{-1}\right)f \tag{7.38}$$

$$= -f\left(\partial_m\frac{1}{2}\mathrm{Tr}\,\left[\overline{\Delta}\Delta\right]\right)f \tag{7.39}$$

$$= -\frac{1}{2}f\mathrm{Tr}\,\left(\overline{\sigma}_m\overline{b}\Delta + \overline{\Delta}b\sigma_m\right)f \tag{7.40}$$

$$= -f\mathrm{Tr}\,\left(\overline{\sigma}_m\overline{b}\Delta\right)f \tag{7.41}$$

$$= -f\mathrm{Tr}\,\left(\overline{\Delta}b\sigma_m\right)f. \tag{7.42}$$

The last two equalities follow from the ADHM equations. After employing all of these results, we have

$$\begin{aligned}D_m\lambda_\alpha = &-\overline{U}b\sigma_m f\overline{\Delta}\mathcal{M}f\overline{b}_\alpha U - \overline{U}\mathcal{M}f\mathrm{Tr}\,\left(\overline{\sigma}_m\overline{b}\Delta\right)f\overline{b}_\alpha U - \overline{U}\mathcal{M}f\overline{b}_\alpha\Delta f\overline{b}\overline{\sigma}_m U\\ &+\overline{U}b_\alpha f\overline{\mathcal{M}}\Delta f\overline{\sigma}_m\overline{b}U + \overline{U}b_\alpha f\mathrm{Tr}\,\left(\overline{\Delta}b\sigma_m\right)f\overline{\mathcal{M}}U + \overline{U}b\sigma_m f\overline{\Delta}b_\alpha f\overline{\mathcal{M}}U.\end{aligned} \tag{7.43}$$

Under contraction with $\overline{\sigma}^{m\dot\alpha\alpha}$, the second and third terms cancel against each other, as do the fifth and sixth. Using equation (A.13), the remainder gives

$$\overline{\sigma}^{m\dot\alpha\alpha}D_m\lambda_\alpha = -2\overline{U}b^\alpha f\left[\overline{\Delta}^{\dot\alpha}\mathcal{M} + \overline{\mathcal{M}}\Delta^{\dot\alpha}\right]f\overline{b}_\alpha U. \tag{7.44}$$

This leads us to the ADHM equation for $\mathcal{M}$,

$$\overline{\Delta}^{\dot\alpha}\mathcal{M} + \overline{\mathcal{M}}\Delta^{\dot\alpha} = 0. \tag{7.45}$$

The constraints on a and b show that this equation is automatically satisfied if $\mathcal{M}$ and $\overline{\mathcal{M}}$ take the values

$$\mathcal{M}_{\beta\lambda}{}^{l} = -4b_{\lambda}{}^{l}{}_{\beta}{}^{\alpha}\xi_{\alpha}, \quad \overline{\mathcal{M}}^{\beta}{}_{l}{}^{\lambda} = -4\xi^{\alpha}\overline{b}_{l}{}^{\lambda}{}_{\alpha}{}^{\beta}, \tag{7.46}$$

or

$$\mathcal{M}_{\beta\lambda}{}^{l} = 4a_{\lambda}{}^{l}{}_{\beta\dot{\alpha}}\overline{\eta}^{\dot{\alpha}}, \quad \overline{\mathcal{M}}^{\beta}{}_{l}{}^{\lambda} = -4\overline{\eta}_{\dot{\alpha}}\overline{a}_{l}{}^{\lambda\dot{\alpha}\beta}, \tag{7.47}$$

which give the supersymmetric and superconformal modes respectively.

We include here two related results. First, there is a reasonably compact expression for $\operatorname{Tr}\lambda\lambda$, using the ADHM data, similar to the corresponding identity for bosonic zero modes due to Corrigan (see appendix B of [28]),

$$\operatorname{Tr}\lambda\lambda = -\frac{1}{4}\partial^{m}\partial_{m}\left[\overline{\mathcal{M}}^{\beta}{}_{l}{}^{\lambda}\left(\delta_{\lambda}{}^{\kappa}\delta_{\beta}{}^{\gamma} + U_{\lambda\beta\dot{\beta}}\overline{U}^{\kappa\dot{\beta}\gamma}\right)\mathcal{M}_{\gamma\kappa}{}^{j}f_{j}{}^{l}\right]. \tag{7.48}$$

Secondly, manipulations like those above can be used to show that the fundamental fermion zero modes, which solve

$$\overline{\sigma}^{m\dot{\alpha}\alpha}D_{m}\psi_{\alpha} = \overline{\sigma}^{m\dot{\alpha}\alpha}\left(\partial_{m}\psi_{\alpha} + v_{m}\psi_{\alpha}\right) = 0, \tag{7.49}$$

are given by

$$(\psi^{\alpha})^{\dot{\beta}} = \overline{U}^{\lambda\dot{\beta}\beta}b_{\lambda}{}^{l}{}_{\beta}{}^{\alpha}f_{l}{}^{j}\mathcal{K}_{j}, \tag{7.50}$$

for any of values of the k Grassmannian parameters $\mathcal{K}_j$. These are the collective coordinates for the k zero modes we expect, from the Atiyah-Singer index theorem.

We shall now turn our attention to the Nahm construction, which will be seen to be closely analogous to the ADHM construction.

7.3 The formalism of the Nahm construction

7.3.1 The gauge field

Monopole configurations do not depend on x^0, and instead the Nahm data involved in the construction of such solutions depend on a reciprocal parameter, s. The foundation of the Nahm construction is therefore the following differential operator, linear in the space variable $x_\mu \sigma^\mu$,

$$\Delta_i^{\ j}{}_{\alpha\dot{\alpha}}(s) = \left(i\frac{d}{ds}\delta_i^{\ j} + (T_0)_i^{\ j}(s)\right)\sigma^0{}_{\alpha\dot{\alpha}} + \left(x_\mu\delta_i^{\ j} + (T_\mu)_i^{\ j}(s)\right)\sigma^\mu{}_{\alpha\dot{\alpha}}, \tag{7.51}$$

where i, j are indices running from 1 to k, the winding number of the monopole. The parameter s is defined on the interval $[-\frac{u}{2}, +\frac{u}{2}]$, where u can be identified with the VEV parameter of the same name introduced previously. Note that this operator already has similar features to the canonical form of the corresponding operator in the ADHM construction; the matrix $T_m\sigma^m$ is the analogue of a in that case. The Hermitian conjugate of Δ is

$$\overline{\Delta}_i^{\ j\dot{\alpha}\alpha}(s) = \left(i\frac{d}{ds}\delta_i^{\ j} + (T_0^\dagger)_i^{\ j}(s)\right)\overline{\sigma}^{0\dot{\alpha}\alpha} + \left(x_\mu\delta_i^{\ j} + (T_\mu^\dagger)_i^{\ j}(s)\right)\overline{\sigma}^{\mu\dot{\alpha}\alpha}. \tag{7.52}$$

We let $\mathcal{U}$ be a vector annihilated by $\overline{\Delta}$, that is

$$\overline{\Delta}_i^{\ j\dot{\alpha}\alpha}(s)\mathcal{U}_{j\alpha}(s) = 0. \tag{7.53}$$

There will be two linearly independent vectors that obey this and can be normalised,

$$\int_{-\frac{u}{2}}^{+\frac{u}{2}} ds\,\overline{\mathcal{U}}^{j\alpha}(s)\mathcal{U}_{j\alpha}(s) < \infty, \tag{7.54}$$

where $\overline{\mathcal{U}}$ is the Hermitian conjugate of $\mathcal{U}$. We can construct out of these vectors a $2k \times 2$ matrix and its Hermitian conjugate,

$$U_{j\alpha\dot{\beta}} = \left(\mathcal{U}_{j\alpha(1)}, \mathcal{U}_{j\alpha(2)}\right), \tag{7.55}$$

$$\overline{U}^{i\dot{\alpha}\alpha} = \begin{pmatrix} \overline{\mathcal{U}}^{i(1)\alpha} \\ \overline{\mathcal{U}}^{i(2)\alpha} \end{pmatrix}, \tag{7.56}$$

which have the following properties;

$$\overline{\Delta}_i^{\ j\dot{\alpha}\alpha}(s)U_{j\alpha\dot{\beta}}(s) = 0, \tag{7.57}$$

$$\left(-i\frac{d\overline{U}}{ds} + \overline{U}T_0\right)\sigma^0 + \overline{U}\left(x_\mu + T_\mu\right)\sigma^\mu = 0, \tag{7.58}$$

and

$$\int_{-\frac{u}{2}}^{+\frac{u}{2}} ds\,\overline{U}^{j\dot{\alpha}\alpha}U_{j\alpha\dot{\beta}} = \delta^{\dot{\alpha}}{}_{\dot{\beta}}. \tag{7.59}$$

The gauge field is given in terms of U and $\overline{U}$ through

$$v_0 = \int_{-\frac{u}{2}}^{+\frac{u}{2}} ds\,\overline{U}(s) is U(s), \tag{7.60}$$

$$v_\mu = \int_{-\frac{u}{2}}^{+\frac{u}{2}} ds\,\overline{U}(s)\partial_\mu U(s). \tag{7.61}$$

(From now on, all integrals over s will be over the range $[-\frac{u}{2}, +\frac{u}{2}]$ unless otherwise specified.)

It will be shown in section 7.3.2 that the Hermitian operator $\overline{\Delta}\Delta$ can be made to factorise so that it is proportional to the 2×2 identity matrix, $\delta^{\dot\beta}{}_{\dot\alpha}$, by imposing certain conditions on the T matrices, in particular they must obey the Nahm equation. This implies that the Green's function for $\overline{\Delta}\Delta$, $f(s,s')$, is also proportional to the 2×2 identity matrix,

$$\left(\overline{\Delta}\Delta\right)_i{}^{j\dot\beta}{}_{\dot\alpha}(s) f_j{}^l(s,s') = \delta(s-s')\delta_i{}^l\delta^{\dot\beta}{}_{\dot\alpha}, \tag{7.62}$$

and it will therefore commute with the sigma matrices σ^m and $\overline{\sigma}^m$.

Now, $U(s)\overline{U}(s')$ and $\Delta(s)f(s,s')\overline{\Delta}(s')$ are both projection operators in the sense that

$$\int ds'\,\mathcal{P}(s,s')\mathcal{P}(s',s'') = \mathcal{P}(s,s''). \tag{7.63}$$

We assume that they obey a completeness relation,

$$U(s)\overline{U}(s') = \delta(s-s') - \Delta(s)f(s,s')\overline{\Delta}(s'). \tag{7.64}$$

This equation should be interpreted as a functional equation, so it applies when multiplied by a suitable function and integrated over s and s'. We now have all the knowledge we need to show that the field strength is self-dual. Using square brackets to denote anti-symmetrisation of indices, we have

$$v_{\mu\nu} = \partial_{[\mu}v_{\nu]} + v_{[\mu}v_{\nu]} \tag{7.65}$$

$$= + \int ds\ \left(\partial_{[\mu}\overline{U}(s)\right)\left(\partial_{\nu]}U(s)\right)$$
$$- \int ds\, ds'\ \left(\partial_{[\mu}\overline{U}(s)\right)U(s)\overline{U}(s')\left(\partial_{\nu]}U(s')\right) \tag{7.66}$$

$$= \int ds\, ds'\ \left(\partial_{[\mu}\overline{U}(s)\right)\Delta(s)f(s,s')\overline{\Delta}(s')\left(\partial_{\nu]}U(s')\right) \tag{7.67}$$

$$= \int ds\, ds'\,\overline{U}(s)\sigma_{[\mu}f(s,s')\overline{\sigma}_{\nu]}U(s') \tag{7.68}$$

$$= 4\int ds\, ds'\,\overline{U}(s)\sigma_{\mu\nu}f(s,s')U(s'), \tag{7.69}$$

where contributions from the end-points after integration by parts will vanish if we impose the boundary conditions $f(s,s') = 0$ for $s = \pm\frac{u}{2}$ on the Green's function. Noting that $\overline{\Delta}(s)[isU(s)] = -\overline{\sigma}^0 U(s)$, we can also find

$$v_{0\mu} = -D_\mu v_0 = -\partial_\mu v_0 - [v_\mu, v_0] \tag{7.70}$$

$$= -\int ds\, ds'\ \left(\partial_\mu\overline{U}(s)\right)\Delta(s)f(s,s')\overline{\Delta}(s')is'U(s')$$
$$- \int ds\, ds'\,\overline{U}(s)is\Delta(s)f(s,s')\overline{\Delta}(s')\left(\partial_\mu U(s')\right) \tag{7.71}$$

$$= -\int ds\, ds'\,\overline{U}(s)\sigma_{[\mu}f(s,s')\overline{\sigma}_{0]}U(s') \tag{7.72}$$

$$= 4\int ds\, ds'\,\overline{U}(s)\sigma_{0\mu}f(s,s')U(s'). \tag{7.73}$$

Therefore,

$$v_{mn} = 4\int ds\, ds'\,\overline{U}(s)\sigma_{mn}f(s,s')U(s'), \tag{7.74}$$

which is obviously self-dual by the self-duality of σ_{mn}.

7.3.2 Factorisation and the Nahm equation

We require that $\overline{\Delta}\Delta$ factorises so that it is proportional to the 2×2 identity matrix, $\delta^{\dot\beta}{}_{\dot\alpha}$. Suppressing indices, we have

$$\overline{\Delta}\Delta = \left(i\frac{d}{ds} + T_0^\dagger\right)\left(i\frac{d}{ds} + T_0\right)\overline{\sigma}^0\sigma^0 + \left(i\frac{d}{ds} + T_0^\dagger\right)(x_\mu + T_\mu)\,\overline{\sigma}^0\sigma^\mu$$
$$+ \left(x_\nu + T_\nu^\dagger\right)\left(i\frac{d}{ds} + T_0\right)\overline{\sigma}^\nu\sigma^0 + \left(x_\nu + T_\nu^\dagger\right)(x_\mu + T_\mu)\,\overline{\sigma}^\nu\sigma^\mu. \tag{7.75}$$

We can use the identity (A.10), and the definition and self-duality of σ^{mn}, to rearrange this into

$$\begin{aligned}
\overline{\Delta}\Delta = &\left[\left(i\frac{d}{ds}+T_0^\dagger\right)\left(i\frac{d}{ds}+T_0\right)+\left(x_\mu+T_\mu^\dagger\right)(x^\mu+T^\mu)\right] \\
&+\left[-\left(T_0-T_0^\dagger\right)x_\mu+(T_\mu-T_\mu^\dagger)i\frac{d}{ds}+x_\nu(T_\rho-T_\rho^\dagger)\epsilon^{\nu\rho\mu}\right]\overline{\sigma}^0\sigma^\mu \\
&+\left[i\frac{dT_\mu}{ds}+T_0^\dagger T_\mu-T_\mu^\dagger T_0+T_\nu^\dagger T_\rho\epsilon^{\nu\rho\mu}\right]\overline{\sigma}^0\sigma^\mu. \qquad (7.76)
\end{aligned}$$

All the terms proportional to $\overline{\sigma}^0\sigma^\mu$ must vanish, and we can consider separately three independent sets,

$$\overline{\sigma}^0\sigma^\mu i\frac{d}{ds} \quad : \quad T_\mu = T_\mu^\dagger, \qquad (7.77)$$

$$\overline{\sigma}^0\sigma^\mu x_\mu \quad : \quad T_0 = T_0^\dagger, \qquad (7.78)$$

$$\overline{\sigma}^0\sigma^\mu \quad : \quad i\frac{dT_\mu}{ds}+[T_0,T_\mu]+\epsilon_{\mu\nu\rho}T_\nu T_\rho = 0. \qquad (7.79)$$

The last condition is the Nahm equation.

7.3.3 Canonical form

Even though the operator $\Delta(s)$ already excludes the (redundant) degrees of freedom represented by the matrix b in the ADHM construction, there are still some symmetries of the Nahm construction remaining that can be used to simplify its form. The gauge field produced by the Nahm construction is invariant under local $U(k)$ transformations,

$$U(s) \mapsto h(s)U(s), \qquad (7.80)$$

$$\overline{U}(s) \mapsto \overline{U}(s)h^\dagger(s), \qquad (7.81)$$

with $h(s) \in U(k)$. All the equations of the Nahm construction are invariant under changes of this form if $f \mapsto hfh^\dagger$, and Δ is a covariant operator, such that for any $\varphi_j{}^{\dot\alpha}(s)$ that transforms as $\varphi \mapsto h\varphi$,

$$\Delta \mapsto \Delta^h, \quad \text{and} \quad \Delta\varphi \mapsto \Delta^h(h\varphi) = h\Delta\varphi. \qquad (7.82)$$

If we write

$$\Delta^h = \left(i\frac{d}{ds}+T_0^h\right)\sigma^0+\left(x_\mu+T_\mu^h\right)\sigma^\mu, \qquad (7.83)$$

then we can deduce the transformation laws for the T matrices;

$$T_\mu^h = hT_\mu h^\dagger, \tag{7.84}$$

$$T_0^h = hT_0 h^\dagger - i\frac{dh}{ds}h^\dagger. \tag{7.85}$$

We may choose

$$h(s) = \lim_{\delta s\to 0} \exp\left\{-iT_0\left(-\frac{u}{2}\right)\delta s\right\}\exp\left\{-iT_0\left(-\frac{u}{2}+\delta s\right)\delta s\right\}\dots$$

$$\dots\exp\Big\{-iT_0\left(s-\delta s\right)\delta s\Big\}\exp\Big\{-iT_0\left(s\right)\delta s\Big\} \tag{7.86}$$

$$\equiv P\exp\left(-i\int_{-\frac{u}{2}}^{s} ds'\,T_0(s')\right), \tag{7.87}$$

where P signifies s ordering, and then

$$T_0^h = 0, \tag{7.88}$$

so we can always make T_0 equal to zero through these s-dependent $U(k)$ transformations. This puts Δ and $\overline{\Delta}$ into the canonical form, where

$$\Delta = i\frac{d}{ds}\sigma^0 + (x_\mu + T_\mu)\,\sigma^\mu, \tag{7.89}$$

$$\overline{\Delta} = i\frac{d}{ds}\overline{\sigma}^0 + (x_\nu + T_\nu)\,\overline{\sigma}^\nu, \tag{7.90}$$

$$\overline{\Delta}\Delta = -\frac{d^2}{ds^2} + (x_\mu + T_\mu)^2\,. \tag{7.91}$$

The Nahm equation becomes

$$i\frac{dT_\mu}{ds} + \epsilon_{\mu\nu\rho}T_\nu T_\rho = 0, \tag{7.92}$$

or

$$i\frac{dT_\mu}{ds} + \frac{1}{2}\epsilon_{\mu\nu\rho}[T_\nu, T_\rho] = 0. \tag{7.93}$$

We shall always use this convenient choice. Global (s-independent) $U(k)$ transformations can still be performed without disturbing the canonical form.

For completeness, we shall briefly discuss here the other transformations relevant to the Nahm construction. It might be thought that local $U(2)$ transformations acting on the Weyl indices, similar to the $U(k)$ group discussed above, would be of interest. However, if such transformations were s-dependent, the factorisation of $\overline{\Delta}\Delta$ would be affected. Global $U(2)$ elements can be used to manipulate Δ, but the $U(1)$ part could be absorbed into a global $U(k)$ transformation, and the remaining $SU(2)$ part is equivalent to a spatial rotation. A more interesting set of alterations is making different choices of the orthonormal basis vectors $\mathcal{U}_{j\alpha(1,2)}$. Since orthonormality, or the condition $\int ds\, \overline{U}^{\dot\alpha} U_{\dot\beta} = \delta^{\dot\alpha}{}_{\dot\beta}$, must be preserved, these changes are equivalent to $U(2)$ transformations acting like

$$U \mapsto U\Omega^\dagger, \tag{7.94}$$

with $\Omega \in U(2)$, where Ω is allowed to depend on the spatial position but not on s. The effect on the gauge field is a local gauge transformation

$$v_0 \mapsto \Omega v_0 \Omega^\dagger, \tag{7.95}$$

$$v_\mu \mapsto \Omega v_\mu \Omega^\dagger + \Omega(\partial_\mu \Omega^\dagger). \tag{7.96}$$

The fact that the group is $U(2)$ shows that the v_m we have constructed here is a $U(2)$ gauge potential. If we are interested in an $SU(2)$ gauge field, we must impose tracelessness on the field through an appropriate $U(2)$ gauge transformation[2].

7.4 One monopole solution

In order to show how monopole configurations are found using the Nahm construction, we shall work through the complete one monopole solution. The first step towards finding a monopole gauge field is to solve the Nahm equation. In the case $k = 1$, the T_μ are simply real functions of s with no matrix structure, so $[T_\nu, T_\rho] = 0$, and the Nahm equation reduces to

$$\frac{dT_\mu}{ds} = 0, \quad \text{or} \quad T_\mu = -X_\mu, \tag{7.97}$$

where the X_μ are constants. Since everything depends on the combination $x + T = x - X$, all positions are naturally measured relative to X, which we can identify as the centre of the monopole configuration and set to be the origin, $X = 0$, by a translation. With all the T_μ equal to zero, the equation $\overline{\Delta}^{\dot\alpha\alpha}\mathcal{U}_\alpha = 0$ becomes

$$\left[i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{d}{ds} - i \begin{pmatrix} x_3 & x_- \\ x_+ & -x_3 \end{pmatrix} \right] \begin{pmatrix} \mathcal{U}_1 \\ \mathcal{U}_2 \end{pmatrix} = 0, \tag{7.98}$$

[2] In fact, the ADHM construction works in this way as well [29].

where $x_\pm = x_1 \pm i x_2$. This pair of ordinary differential equations are easily solved;

$$\mathcal{U}_{1(1)} = A_1 x_- e^{|x|s} + B_1 x_- e^{-|x|s}, \tag{7.99}$$

$$\mathcal{U}_{2(1)} = A_1(|x| - x_3)e^{|x|s} - B_1(|x| + x_3)e^{-|x|s}, \tag{7.100}$$

where $|x| = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$, and similarly for $\mathcal{U}_{(2)}$, with A_1, A_2, B_1 and B_2 determined by the condition of orthonormality. For convenience, we will use a $U(2)$ gauge transformation to rotate U into the simple form,

$$U = \begin{pmatrix} A x_- e^{|x|s} & B x_- e^{-|x|s} \\ A(|x| - x_3)e^{|x|s} & -B(|x| + x_3)e^{-|x|s} \end{pmatrix}, \tag{7.101}$$

with A and B real. Then the normalisation condition $\int ds\, \overline{U}^{\dot\alpha} U_{\dot\beta} = \delta^{\dot\alpha}{}_{\dot\beta}$ implies

$$A = (2(|x| - x_3)\sinh u|x|)^{-\frac{1}{2}} = (2|x|(|x| - x_3))^{-\frac{1}{2}}\alpha, \tag{7.102}$$

$$B = (2(|x| + x_3)\sinh u|x|)^{-\frac{1}{2}} = (2|x|(|x| + x_3))^{-\frac{1}{2}}\alpha, \tag{7.103}$$

with $\alpha^{-2} \equiv \frac{\sinh u|x|}{|x|}$. For the remainder of this section, the symbol U will refer to this matrix.

7.4.1 The gauge field component v_0

We can now calculate the gauge field using the Nahm construction expressions, and we shall consider first the zeroth component,

$$v_0 = \int_{-\frac{u}{2}}^{+\frac{u}{2}} ds\, \overline{U} i s U \tag{7.104}$$

$$= i\alpha^2 \begin{pmatrix} \int ds\, s e^{2|x|s} & 0 \\ 0 & \int ds\, s e^{-2|x|s} \end{pmatrix} \tag{7.105}$$

$$= -\frac{\tau_3}{2i}\left(u \coth u|x| - \frac{1}{|x|}\right). \tag{7.106}$$

Obviously, we have chosen a singular gauge, where the asymptotic value of v_0 as $|x| \to \infty$ is in the τ_3 direction. In order to move to a regular gauge where v_0 tends to something proportional to $x^a \tau_a$, we can apply an $SU(2)$ transformation, $v_0 \mapsto v_0' = \Omega v_0 \Omega^\dagger$, or $U \mapsto U' = U\Omega^\dagger$, where

$$\Omega = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \tag{7.107}$$

with $\bar{a}a + \bar{b}b = 1$, and

$$2\bar{a}a = \frac{|x| + x_3}{|x|}, \tag{7.108}$$

$$2\bar{b}b = \frac{|x| - x_3}{|x|}, \tag{7.109}$$

$$-2ab = \frac{x_1 - ix_2}{|x|}. \tag{7.110}$$

Note that the absolute phase of a or b is not determined, only the phase of ab. This corresponds to the fact that there is a $U(1)$ subgroup of the $SU(2)$ gauge group that leaves the asymptotic value of v_0 unaltered,

$$v_0 = e^{-\Omega\frac{\tau_3}{2i}} v_0 e^{\Omega\frac{\tau_3}{2i}}. \tag{7.111}$$

We shall make the definite choice,

$$a = \left(\frac{|x| + x_3}{2|x|}\right)^{\frac{1}{2}}, \quad b = \frac{-x_-}{(2|x|(|x| + x_3))^{\frac{1}{2}}}, \tag{7.112}$$

then we can find the new matrix $U' = U\Omega^\dagger$,

$$(U')_{11} = x_-(A\bar{a}e^{|x|s} + B\bar{b}e^{-|x|s}), \tag{7.113}$$

$$(U')_{12} = x_-(-Abe^{|x|s} + Bae^{-|x|s}), \tag{7.114}$$

$$(U')_{21} = A\bar{a}(|x| - x_3)e^{|x|s} - B\bar{b}(|x| + x_3)e^{-|x|s}, \tag{7.115}$$

$$(U')_{22} = -Ab(|x| - x_3)e^{|x|s} - Ba(|x| + x_3)e^{-|x|s}. \tag{7.116}$$

and use it in the Nahm construction to find the transformed gauge potential. It does indeed give

$$v_0' = i\frac{x^\mu \tau_\mu}{|x|}\alpha^2 \int_{-\frac{u}{2}}^{+\frac{u}{2}} ds\, s \sinh 2|x|s \tag{7.117}$$

$$= -\frac{x^\mu \tau_\mu}{|x|2i}\left(u \coth u|x| - \frac{1}{|x|}\right), \tag{7.118}$$

which we recognise as the standard form of v_0 for the BPS monopole, from section 4.2.1.

7.4.2 The gauge field components v_μ

The fields v_1, v_2, and v_3 are treated differently to v_0 in the Nahm construction. We shall begin by calculating the first component of the gauge field,

$$v_1 = \int_{-\frac{u}{2}}^{+\frac{u}{2}} ds\, \overline{U}\partial_1 U. \tag{7.119}$$

With U as above, we have

$$\partial_1 U = \begin{pmatrix} PAx_- e^{|x|s} & QBx_- e^{-|x|s} \\ RA(|x| - x_3)e^{|x|s} & -SB(|x| + x_3)e^{-|x|s} \end{pmatrix}, \tag{7.120}$$

where

$$P = \frac{\partial_1 A}{A} + \frac{1}{x_-} + \frac{x_1 s}{|x|}, \qquad Q = \frac{\partial_1 B}{B} + \frac{1}{x_-} - \frac{x_1 s}{|x|}, \tag{7.121}$$

$$R = \frac{\partial_1 A}{A} + \frac{x_1}{|x|(|x| - x_3)} + \frac{x_1 s}{|x|}, \quad S = \frac{\partial_1 B}{B} + \frac{x_1}{|x|(|x| + x_3)} - \frac{x_1 s}{|x|}. \tag{7.122}$$

This leads to

$$v_1 = \begin{pmatrix} \frac{ix_2}{2|x|(|x|-x_3)} & AB\left(ix_2 + \frac{x_1x_3}{|x|}\right)u \\ AB\left(ix_2 - \frac{x_1x_3}{|x|}\right)u & \frac{ix_2}{2|x|(|x|+x_3)} \end{pmatrix}, \tag{7.123}$$

where we note that $AB = \left[2(|x|^2 - x_3^2)^{\frac{1}{2}} \sinh u|x|\right]^{-1}$. Clearly v_1 is not traceless, so our simple choice of U does not lead to an acceptable $SU(2)$ potential. We must use the $U(2)$ gauge freedom to remove the trace, but without affecting the v_0 component since it gave the correct monopole field. Any $\omega \in U(2)$ of the form $\omega = \exp[i(\chi + \xi\tau_3)]$ sends $v_0 \mapsto \omega v_0 \omega^\dagger = v_0$, but transforms v_1 to $v_1'' = \omega v_1 \omega^\dagger + \omega\partial_1\omega^\dagger$,

$$v_1'' = \begin{pmatrix} (v_1)^1{}_1 - i\partial_1\chi - i\partial_1\xi & (v_1)^1{}_2 e^{2i\xi} \\ (v_1)^2{}_1 e^{-2i\xi} & (v_1)^2{}_2 - i\partial_1\chi + i\partial_1\xi \end{pmatrix}. \tag{7.124}$$

Let us take $\chi = \xi$, so we leave $(v_1)^2{}_2$ unchanged and must arrange $(v_1'')^1{}_1 = -(v_1)^2{}_2$. Therefore, we require $2\partial_1\xi = \frac{x_2}{x_1^2+x_2^2}$. Integrating gives $\xi = \frac{1}{2}\arg(x_-) + C$, with a constant (or function of x_2 and x_3), C, which can be chosen such that

$$e^{2i\xi} = -\frac{x_-}{(|x|^2 - x_3^2)^{\frac{1}{2}}}, \quad \omega = \begin{pmatrix} -\frac{x_-}{|x_-|} & 0 \\ 0 & 1 \end{pmatrix}. \tag{7.125}$$

Then, the $SU(2)$ gauge field component becomes

$$v_1'' = \begin{pmatrix} \dfrac{-ix_2}{2|x|(|x| + x_3)} & \dfrac{-(ix_2|x| + x_1x_3)\,ux_-}{2|x|(|x|^2 - x_3^2)\sinh u|x|} \\ \dfrac{-(ix_2|x| - x_1x_3)\,ux_+}{2|x|(|x|^2 - x_3^2)\sinh u|x|} & \dfrac{+ix_2}{2|x|(|x| + x_3)} \end{pmatrix}. \tag{7.126}$$

We can make a comparison to the known expression, from section 4.2.1,

$$\hat{v}_\mu = -i\epsilon_{a\mu\nu}\frac{x^\nu\tau^a}{2|x|^2}\left(1 - \frac{u|x|}{\sinh u|x|}\right), \tag{7.127}$$

$$\hat{v}_1 = -\frac{1}{2|x|^2}\left(1 - \frac{u|x|}{\sinh u|x|}\right)\begin{pmatrix} ix_2 & -x_3 \\ x_3 & -ix_2 \end{pmatrix}, \tag{7.128}$$

by using the transformation $\Omega^\dagger$ to bring this into a singular gauge with v_0 proportional to τ_3. So, if we transform to $\check{v}_\mu = \Omega^\dagger \hat{v}_\mu \Omega + \Omega^\dagger \partial_\mu \Omega$, we expect $\check{v}_1$ to be comparable to v_1''. Writing

$$\Omega = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} = (2|x|(|x| + x_3))^{-\frac{1}{2}}\begin{pmatrix} |x| + x_3 & -x_1 + ix_2 \\ x_1 + ix_2 & |x| + x_3 \end{pmatrix}, \tag{7.129}$$

we have

$$\Omega^\dagger \hat{v}_1 \Omega = -\frac{1}{2|x|^2(|x| + x_3)}\left(1 - \frac{u|x|}{\sinh u|x|}\right)$$
$$\begin{pmatrix} 0 & -x_3(|x| + x_3) - ix_2 x_- \\ x_3(|x| + x_3) - ix_2 x_+ & 0 \end{pmatrix}, \tag{7.130}$$

and if we set $\varphi = (2|x|(|x| + x_3))^{-\frac{1}{2}}$, then

$$\partial_1 \Omega = \varphi^{-1}(\partial_1 \varphi)\Omega + \varphi \begin{pmatrix} \frac{x_1}{|x|} & -1 \\ 1 & \frac{x_1}{|x|} \end{pmatrix}, \tag{7.131}$$

with $\partial_1 \varphi = -\varphi^3 \frac{x_1}{|x|}(2|x| + x_3)$, so

$$\Omega^\dagger(\partial_1 \Omega) = \frac{1}{2|x|(|x| + x_3)}\begin{pmatrix} -ix_2 & \frac{x_1 x_-}{|x|} - |x| - x_3 \\ -\frac{x_1 x_+}{|x|} + |x| + x_3 & ix_2 \end{pmatrix}. \tag{7.132}$$

Finally, therefore, we arrive at

$$\check{v}_1 = \begin{pmatrix} \dfrac{-ix_2}{2|x|(|x| + x_3)} & \dfrac{-(ix_2|x| + x_1 x_3)\, u x_-}{2|x|(|x|^2 - x_3^2)\sinh u|x|} \\ \dfrac{-(ix_2|x| - x_1 x_3)\, u x_+}{2|x|(|x|^2 - x_3^2)\sinh u|x|} & \dfrac{+ix_2}{2|x|(|x| + x_3)} \end{pmatrix}, \tag{7.133}$$

which is exactly v_1''. The various gauges used here, and the connections between them, are summarised in Table 7.1.

The same procedure has been applied in order to determine and check the remaining two components, and the results are shown in appendix D.

	Singular gauges		Regular gauges
	$v_0 \to -\frac{\tau_3}{2i}u$		$v_0 \to -\frac{\tau_a x^a}{2i\lvert x\rvert}u$
$U(2)$	v	$\xrightarrow{\Omega}$	v'
	$\downarrow \omega$		
$SU(2)$	$v'' = \check{v}$	$\xleftarrow{\Omega^\dagger}$	$\hat{v}$

(7.134)

Table 7.1: Properties of the different gauges used in the calculation of the one monopole gauge field.

7.4.3 The field strength

The field strength v_{mn} would normally be obtained from the potential v_m with the defining equation,

$$v_{mn} = \partial_m v_n - \partial_n v_m + [v_m, v_n], \tag{7.135}$$

but in the case of a self-dual monopole solution, the Nahm construction allows the field strength to be found directly using

$$v_{mn} = 4\int ds\, ds'\, \overline{U}(s)\sigma_{mn} f(s,s') U(s'). \tag{7.136}$$

However, the derivation of this formula used the completeness relation, which was postulated rather then proved. Reproducing the one monopole field strength from this equation will therefore be a check of our assumption.

We know the matrices U and $\overline{U}$ in the one monopole case, and in order to apply this formula we must also find f, the Green's function for the operator

$$\overline{\Delta}\Delta = -\frac{d^2}{ds^2} + |x|^2, \tag{7.137}$$

which has the boundary conditions $f(s,s') = 0$ for $s = \pm\frac{u}{2}$. It is

$$f(s,s') = \begin{cases} -\dfrac{\sinh|x|\left(s+\frac{u}{2}\right)\sinh|x|\left(s'-\frac{u}{2}\right)}{|x|\sinh(u|x|)} & s < s' \\ -\dfrac{\sinh|x|\left(s-\frac{u}{2}\right)\sinh|x|\left(s'+\frac{u}{2}\right)}{|x|\sinh(u|x|)} & s > s' \end{cases} \tag{7.138}$$

$$\begin{aligned} &= -\theta(s'-s)\frac{\sinh|x|\left(s+\frac{u}{2}\right)\sinh|x|\left(s'-\frac{u}{2}\right)}{|x|\sinh(u|x|)} \\ &\quad -\theta(s-s')\frac{\sinh|x|\left(s-\frac{u}{2}\right)\sinh|x|\left(s'+\frac{u}{2}\right)}{|x|\sinh(u|x|)}. \end{aligned} \tag{7.139}$$

The same function, written slightly differently, was given by Nahm in [77]. It has the correct behaviour at the end-points of the interval $[-\frac{u}{2}, +\frac{u}{2}]$, and using $\frac{d}{ds}\theta(s-s') = \delta(s-s')$ and $\delta(s-s')F(s) = \delta(s-s')F(s')$, for any function F, we can also see that

$$\begin{aligned}\frac{df}{ds} =& -\theta(s'-s)|x|\frac{\cosh|x|\left(s+\frac{u}{2}\right)\sinh|x|\left(s'-\frac{u}{2}\right)}{|x|\sinh(u|x|)}\\ &-\theta(s-s')|x|\frac{\cosh|x|\left(s-\frac{u}{2}\right)\sinh|x|\left(s'+\frac{u}{2}\right)}{|x|\sinh(u|x|)},\end{aligned} \tag{7.140}$$

and similarly,

$$\begin{aligned}\frac{d^2f}{ds^2} =& -\theta(s'-s)|x|^2\frac{\sinh|x|\left(s+\frac{u}{2}\right)\sinh|x|\left(s'-\frac{u}{2}\right)}{|x|\sinh(u|x|)}\\ &-\theta(s-s')|x|^2\frac{\sinh|x|\left(s-\frac{u}{2}\right)\sinh|x|\left(s'+\frac{u}{2}\right)}{|x|\sinh(u|x|)}\\ &+\delta(s-s')\frac{\cosh|x|\left(s+\frac{u}{2}\right)\sinh|x|\left(s-\frac{u}{2}\right)}{\sinh(u|x|)}\\ &-\delta(s-s')\frac{\cosh|x|\left(s-\frac{u}{2}\right)\sinh|x|\left(s+\frac{u}{2}\right)}{\sinh(u|x|)}\end{aligned} \tag{7.141}$$

$$= -\delta(s-s') + |x|^2 f(s,s'). \tag{7.142}$$

Note that f has the following properties: $f(s,s') = f(s',s)$, as expected for the Green's function of a Hermitian operator; and $f(-s,-s') = f(s,s')$, which occurs because $\overline{\Delta}\Delta(-s) = \overline{\Delta}\Delta(s)$ and $\delta(s-s') = \delta(s'-s)$.

We are now able to calculate the one monopole field strength, and we will consider the v_{12} component. The Green's function is a real function with no indices, therefore it commutes with all matrices. Using $\sigma_{12} = \frac{i}{2}\tau_3$, we have

$$v_{12} = 2i\int ds\, ds'\, \overline{U}(s)\tau_3 U(s') f(s,s'). \tag{7.143}$$

The combination $\overline{U}(s)\tau_3 U(s')$ is easily found,

$$\overline{U}(s)\tau_3 U(s') = \frac{\alpha^2}{|x|}\begin{pmatrix} x_3 e^{|x|(s+s')} & (|x|^2 - x_3^2)^{\frac{1}{2}} e^{|x|(s-s')} \\ (|x|^2 - x_3^2)^{\frac{1}{2}} e^{|x|(s'-s)} & -x_3 e^{-|x|(s+s')} \end{pmatrix}. \tag{7.144}$$

We can see that the field strength will be traceless by changing variables to $z = -s$ and

$z' = -s'$ and using the identity $f(s,s') = f(z,z')$,

$$(v_{12})^1{}_1 = 2i \int_{-\frac{u}{2}}^{+\frac{u}{2}} ds \int_{-\frac{u}{2}}^{+\frac{u}{2}} ds' \alpha^2 \frac{x_3}{|x|} e^{|x|(s+s')} f(s,s') \tag{7.145}$$

$$= 2i \int_{-\frac{u}{2}}^{+\frac{u}{2}} dz \int_{-\frac{u}{2}}^{+\frac{u}{2}} dz' \alpha^2 \frac{x_3}{|x|} e^{-|x|(z+z')} f(z,z') \tag{7.146}$$

$$= -(v_{12})^2{}_2. \tag{7.147}$$

The Green's function f has a different form for $s > s'$ and $s < s'$, so the integral naturally splits into two parts,

$$\begin{aligned} v_{12} = & 2i \int_{-\frac{u}{2}}^{+\frac{u}{2}} ds \int_{-\frac{u}{2}}^{s} ds' \overline{U}(s) \tau_3 U(s') \left[f(s,s')|_{s>s'} \right] \\ & + 2i \int_{-\frac{u}{2}}^{+\frac{u}{2}} ds' \int_{-\frac{u}{2}}^{s'} ds \, \overline{U}(s) \tau_3 U(s') \left[f(s,s')|_{s<s'} \right]. \end{aligned} \tag{7.148}$$

If we exchange the dummy variable labels s and s' in the second integral and use $f(s,s') = f(s',s)$, then this becomes

$$v_{12} = 2i \int_{-\frac{u}{2}}^{+\frac{u}{2}} ds \int_{-\frac{u}{2}}^{s} ds' \left\{ \overline{U}(s) \tau_3 U(s') + \overline{U}(s') \tau_3 U(s) \right\} \left[f(s,s')|_{s>s'} \right]. \tag{7.149}$$

Now,

$$\begin{aligned} & \overline{U}(s)\tau_3 U(s') + \overline{U}(s')\tau_3 U(s) = \\ & \frac{\alpha^2}{|x|} \begin{pmatrix} 2x_3 e^{|x|(s+s')} & 2(|x|^2 - x_3^2)^{\frac{1}{2}} \cosh|x|(s-s') \\ 2(|x|^2 - x_3^2)^{\frac{1}{2}} \cosh|x|(s'-s) & -2x_3 e^{-|x|(s+s')} \end{pmatrix}, \end{aligned} \tag{7.150}$$

and the equality of the off-diagonal elements in this formula implies that $(v_{12})^1{}_2 = (v_{12})^2{}_1$, which is consistent with the field strength being anti-Hermitian as all the components of v_{12} are imaginary in this gauge. It is now sufficient to calculate just two of the components of v_{12}, one diagonal and one off-diagonal, and use tracelessness and anti-Hermitianality to find the remaining two. Using hyperbolic trigonometric function identities, we can find

$$\begin{aligned} (v_{12})^1{}_2 = & -\frac{4i(|x|^2 - x_3^2)^{\frac{1}{2}}}{|x| \sinh^2 u|x|} \int_{-\frac{u}{2}}^{+\frac{u}{2}} ds \sinh\left(|x|\left(s - \frac{u}{2}\right)\right) \\ & \left\{ \int_{-\frac{u}{2}}^{s} ds' \cosh(|x|(s'-s)) \sinh\left(|x|\left(s' + \frac{u}{2}\right)\right) \right\} \end{aligned} \tag{7.151}$$

$$= -\frac{i(|x|^2 - x_3^2)^{\frac{1}{2}} u}{2|x|^2 \sinh^2 u|x|} \left(\sinh(u|x|) - u|x| \cosh(u|x|)\right), \tag{7.152}$$

$$(v_{12})^1{}_1 = -\frac{4ix_3}{|x|\sinh^2(u|x|)}\int_{-\frac{u}{2}}^{+\frac{u}{2}} ds \sinh\left(|x|\left(s-\frac{u}{2}\right)\right)$$
$$\left\{\int_{-\frac{u}{2}}^{s} ds'\, e^{|x|(s+s')}\sinh\left(|x|\left(s'+\frac{u}{2}\right)\right)\right\} \tag{7.153}$$
$$= \frac{ix_3}{2|x|\sinh^2(u|x|)}\left\{\frac{1}{|x|^2}\sinh^2(u|x|) - u^2\right\}. \tag{7.154}$$

The gauge transformation ω sends $v_{mn} \mapsto v''_{mn} = \omega v_{mn}\omega^\dagger$, where v''_{mn} is the field strength associated with the traceless gauge potential v''_m;

$$(v''_{12})^1{}_1 = -(v''_{12})^2{}_2 = \frac{ix_3}{2|x|\sinh^2(u|x|)}\left\{\frac{1}{|x|^2}\sinh^2(u|x|) - u^2\right\}, \tag{7.155}$$

$$(v''_{12})^1{}_2 = \frac{iux_-}{2|x|^2\sinh^2 u|x|}\left(\sinh(u|x|) - u|x|\cosh(u|x|)\right), \tag{7.156}$$

$$(v''_{12})^2{}_1 = \frac{iux_+}{2|x|^2\sinh^2 u|x|}\left(\sinh(u|x|) - u|x|\cosh(u|x|)\right). \tag{7.157}$$

We can check this by taking the field strength calculated in section 4.2.1, which has the form

$$\hat{v}_{12} = \frac{\tau_3}{2i|x|^2}\frac{u|x|}{\sinh(u|x|)}\left(1 - u|x|\coth(u|x|)\right)$$
$$+\frac{x_3}{2i|x|^4}\left[\frac{u|x|}{\sinh(u|x|)}\left(\frac{u|x|}{\sinh(u|x|)} + u|x|\coth(u|x|) - 1\right) - 1\right]\begin{pmatrix} x_3 & x_- \\ x_+ & -x_3\end{pmatrix}, \tag{7.158}$$

and transforming it to $\check{v}_{12} = \Omega^\dagger\hat{v}_{12}\Omega$, which should be the same as v''_{12}. By design,

$$\Omega^\dagger\begin{pmatrix} x_3 & x_- \\ x_+ & -x_3\end{pmatrix}\Omega = |x|\tau_3, \tag{7.159}$$

and matrix multiplication gives

$$\Omega^\dagger\tau_3\Omega = \frac{1}{|x|}\begin{pmatrix} x_3 & -x_- \\ -x_+ & -x_3\end{pmatrix}. \tag{7.160}$$

Therefore,

$$(\check{v}_{12})^1{}_1 = -(\check{v}_{12})^2{}_2 = \frac{ix_3}{2|x|\sinh^2(u|x|)}\left\{\frac{1}{|x|^2}\sinh^2(u|x|) - u^2\right\} = (v''_{12})^1{}_1, \tag{7.161}$$

$$(\check{v}_{12})^1{}_2 = \frac{iux_-}{2|x|^2 \sinh^2 u|x|} \left(\sinh(u|x|) - u|x| \cosh(u|x|)\right) = (v''_{12})^1{}_2, \tag{7.162}$$

$$(\check{v}_{12})^2{}_1 = \frac{iux_+}{2|x|^2 \sinh^2 u|x|} \left(\sinh(u|x|) - u|x| \cosh(u|x|)\right) = (v''_{12})^2{}_1. \tag{7.163}$$

The calculation of the remaining field strength components, v_{23} and v_{31}, proceeds analogously to the case of v_{12}. The results are summarised in appendix D, and there is full agreement between the field strength calculated from the Nahm construction and the known expressions for the BPS field strength.

7.5 Properties of the Green's function

We shall now prove two identities that are necessary for the determination of the adjoint fermion zero modes of a monopole in the Nahm formalism. They are also useful in showing that the equations of motion $D^m v_{mn} = 0$ are indeed satisfied by the Nahm construction gauge field (and field strength). This is guaranteed by self-duality, but is another successful test of the field strength formula, equation (7.74), which rests on the assumed completeness relation.

The identities are relevant to the Green's function of the operator $\overline{\Delta}\Delta$, f. One is a formula for its spatial derivative, $\partial_\mu f$, similar to the expression for $\partial_m f$ in the ADHM construction. The other is an identity that substitutes for the missing component $\partial_0 f$, with an intuitive form if s is viewed as the reciprocal variable of x^0. It has been checked that the one monopole version of f, equation (7.139), obeys both identities.

Recall that the Green's function we are interested in is the solution of the equation

$$(\overline{\Delta}\Delta)_i{}^{j\dot\beta}{}_{\dot\alpha}(s) f_j{}^l(s,s') = \delta(s-s')\delta_i{}^l \delta^{\dot\beta}{}_{\dot\alpha}, \tag{7.164}$$

with the boundary conditions that $f(s,s') = 0$ for $s = \pm\frac{u}{2}$, and also, that any equation of the form

$$\overline{\Delta}\Delta(s)\phi(s,s') = \mathcal{J}(s,s'), \tag{7.165}$$

where ϕ has the same boundary conditions as f, has the solution

$$\phi(s,s') = \int ds''\, f(s,s'')\mathcal{J}(s'',s'). \tag{7.166}$$

In order to find the gradient of f, we can differentiate the equation

$$\frac{1}{2}\mathrm{Tr}\left(\overline{\Delta}\Delta(s)\right) f(s,s') = \delta(s-s'), \tag{7.167}$$

with respect to x^μ, and we find

$$\frac{1}{2}\mathrm{Tr}\left(\overline{\Delta}\Delta(s)\right)\left(\partial_\mu f(s,s')\right) = -\frac{1}{2}\left[\partial_\mu \mathrm{Tr}\left(\overline{\Delta}\Delta(s)\right)\right] f(s,s'). \tag{7.168}$$

Therefore,

$$\begin{aligned}\partial_\mu f(s,s') &= -\frac{1}{2}\int ds''\, f(s,s'')\,\mathrm{Tr}\left[\partial_\mu\left(\overline{\Delta}\Delta\right)(s'')\right] f(s'',s') &(7.169)\\ &= -\int ds''\, f(s,s'')\,\mathrm{Tr}\left[\overline{\sigma}_\mu\Delta(s'')\right] f(s'',s'). &(7.170)\end{aligned}$$

To reach the second identity, we act on the function $i(s-s')f(s,s')$ with $\overline{\Delta}\Delta$, which gives

$$\begin{aligned}\left[\overline{\Delta}\Delta(s)\right] i(s-s')f(s,s') = &-\frac{d}{ds}\left[if(s,s') + i(s-s')\frac{df}{ds}(s,s')\right]\\ &+ i(s-s')\left[x+T\right]^2 f(s,s') &(7.171)\\ = &-2i\frac{df}{ds}(s,s') + i(s-s')\delta(s-s') &(7.172)\\ = &-2i\frac{df}{ds}(s,s'). &(7.173)\end{aligned}$$

Therefore we have the identity

$$\begin{aligned}i(s-s')f(s,s') &= -2\int ds''\, f(s,s'')\, i\frac{df}{ds''}(s'',s') &(7.174)\\ &= -\int ds''\, f(s,s'')\,\mathrm{Tr}\left[\overline{\sigma}_0\Delta(s'')\right] f(s'',s'). &(7.175)\end{aligned}$$

7.6 Adjoint fermion zero modes

A fermion zero mode in the adjoint representation is a solution of the Dirac equation,

$$\overline{\sigma}^{m\dot{\alpha}\alpha} D_m\lambda_\alpha = \overline{\sigma}^{m\dot{\alpha}\alpha}\left(\partial_m\lambda_\alpha + [v_m,\lambda_\alpha]\right) = 0. \tag{7.176}$$

Following the analogy with the ADHM construction, we shall assume the following expression for a zero mode,

$$\begin{aligned}(\lambda_\alpha)^{\dot{\beta}}{}_{\dot{\gamma}} = &+\int ds\,ds'\,\overline{U}^{i\dot{\beta}\gamma}(s)\mathcal{M}_{\gamma i}{}^{j}(s) f_j{}^{l}(s,s')U_{l\alpha\dot{\gamma}}(s')\\ &-\int ds\,ds'\,\overline{U}^{i\dot{\beta}}{}_{\alpha}(s) f_i{}^{j}(s,s')\overline{\mathcal{M}}^{\gamma}{}_{j}{}^{l}(s')U_{l\gamma\dot{\gamma}}(s'), &(7.177)\end{aligned}$$

where $\mathcal{M}_\gamma$ and $\overline{\mathcal{M}}^\gamma$ are Grassmannian matrices. A similar procedure to that used in the ADHM case, including the application of equation (7.175), shows that in order for the form above to be a solution, $\mathcal{M}_\gamma$ and $\overline{\mathcal{M}}^\gamma$ must obey the following constraint, reminiscent of equation (7.45),

$$\overline{\Delta}_i^{\ j\dot{\alpha}\gamma}\mathcal{M}_{\gamma j}^{\ \ l} + \overline{\mathcal{M}}_{\gamma i}^{\ \ j}\Delta_{j\ \gamma}^{\ l\ \dot{\alpha}} = 0. \tag{7.178}$$

The part of this proportional to x_μ should vanish independently, which implies

$$\mathcal{M}_{\gamma i}^{\ \ j} = \overline{\mathcal{M}}_{\gamma i}^{\ \ j}. \tag{7.179}$$

The remainder leads to

$$\overline{\sigma}^{0\dot{\alpha}\gamma} i\frac{d}{ds}\mathcal{M}_{\gamma i}^{\ \ l} + \overline{\sigma}^{\mu\dot{\alpha}\gamma}[T_\mu, \mathcal{M}_\gamma]_i^{\ l} = 0. \tag{7.180}$$

We shall discuss here two interesting solutions to this equation, and therefore two types of adjoint fermion zero modes. The first gives the supersymmetric zero modes, which are already well known. Every monopole solution has these two zero modes, and for the one monopole they are the only adjoint fermion zero modes. The second type is two further zero modes that are possessed by every monopole solution with $k > 1$, and that also have an easy interpretation. Altogether, these four zero modes give us all of the two monopole zero modes, which would be sufficient for the calculation of the four-fermion correlation function, equation (7.1), that we were hoping to achieve.

We might also mention here that an attempt was made to find a formula for the square of the most general monopole zero mode, Tr $\lambda\lambda$, similar to equation (7.48). However, the presence of the extra parameter, s, meant that progress was much harder, and also that any eventual result was unlikely to be a total derivative, or therefore as useful as the corresponding ADHM version. Consequently, this approach was abandoned.

7.6.1 The supersymmetric mode

If we look for s-independent solutions to equation (7.180), obeying

$$\overline{\sigma}^{\mu\dot{\alpha}\gamma}[T_\mu, \mathcal{M}_\gamma] = 0, \tag{7.181}$$

then because the T_μ do vary with s, $\mathcal{M}_\gamma$ must commute with each of them individually,

$$[T_\mu, \mathcal{M}_\gamma] = 0. \tag{7.182}$$

This will certainly be true if

$$\mathcal{M}_{\gamma j}{}^{l} = -4\xi_\gamma \delta_j{}^l, \tag{7.183}$$

which is the value that gives the supersymmetric zero mode. Furthermore, at $s = \pm\frac{u}{2}$, the appropriate boundary conditions [76] for the T matrices are that they have simple poles, with residues that form a set of generators of the k dimensional irreducible representation of $SU(2)$. Therefore, any constant $\mathcal{M}_\gamma$ has to commute with all of these, and so by Schur's lemma, having $\mathcal{M}_\gamma$ proportional to the $k \times k$ identity matrix is the only s-independent solution.

7.6.2 The dual supersymmetric mode

In order to find a new solution (with $\mathcal{M}_\gamma$ depending on s), we shall now discuss a different interpretation of the Nahm equation. Consider a Hermitian $U(k)$ gauge potential, T_m, which is only dependent on one coordinate[3], s, and in the gauge where the corresponding field component, say T_0, is zero. Then the field strength,

$$T_{mn} = \partial_m T_n - \partial_n T_m - i[T_m, T_n], \tag{7.184}$$

reduces to

$$T_{0\rho} = \frac{dT_\rho}{ds}, \tag{7.185}$$

and

$$T_{\mu\nu} = -i[T_\mu, T_\nu], \tag{7.186}$$

and requiring this to be self-dual leads directly to the Nahm equation. Also, the Dirac equation for the adjoint representation (of $U(k)$) is exactly the constraint on $\mathcal{M}_\gamma$ above, and we can immediately write down one solution, the supersymmetric zero mode in this $U(k)$ theory;

$$\mathcal{M}_{\gamma i}{}^{j} = -\frac{1}{2}\sigma^{mn}{}_{\gamma}{}^{\beta}\eta_\beta (T_{mn})_i{}^j, \tag{7.187}$$

or

$$\mathcal{M}^{\gamma}{}_{i}{}^{j} = \frac{1}{2}\eta^\beta \sigma^{mn}{}_{\beta}{}^{\gamma}(T_{mn})_i{}^j. \tag{7.188}$$

[3]In comparison, the monopole gauge field v_m depends on three out of four space coordinates.

By the self-duality of T_{mn}, this can also be written

$$\mathcal{M}^{\gamma}{}_i^j = -i\eta^\beta \sigma^{\mu\nu}{}_\beta{}^\gamma [T_\mu, T_\nu]_i^j, \tag{7.189}$$

or

$$\mathcal{M}^{\gamma}{}_i^j = 2\eta^\beta \sigma^{0\rho}{}_\beta{}^\gamma \frac{d}{ds} (T_\rho)_i^j. \tag{7.190}$$

To check that this is a solution to the constraint, note that

$$\overline{\sigma}^{0\dot{\alpha}\gamma} i \frac{d\mathcal{M}_\gamma}{ds} = -(\overline{\sigma}^0 \sigma^{\mu\nu}\eta)^{\dot{\alpha}} \frac{d}{ds}[T_\mu, T_\nu] \tag{7.191}$$

$$= -2(\overline{\sigma}^0 \sigma^{\mu\nu}\eta)^{\dot{\alpha}} \left[\frac{dT_\mu}{ds}, T_\nu\right], \tag{7.192}$$

and

$$\overline{\sigma}^{\rho\dot{\alpha}\gamma}[T_\rho, \mathcal{M}_\gamma] = -2(\overline{\sigma}^\rho \sigma^{0\sigma}\eta)^{\dot{\alpha}} \left[T_\rho, \frac{dT_\sigma}{ds}\right], \tag{7.193}$$

so

$$\overline{\sigma}^{0\dot{\alpha}\gamma} i \frac{d\mathcal{M}_\gamma}{ds} + \overline{\sigma}^{\rho\dot{\alpha}\gamma}[T_\rho, \mathcal{M}_\gamma] = -2(\overline{\sigma}^0 \sigma^{\mu\nu} - \overline{\sigma}^\nu \sigma^{0\mu})^{\dot{\alpha}\beta} \eta_\beta \left[\frac{dT_\mu}{ds}, T_\nu\right]. \tag{7.194}$$

Now, using identities from appendix A, we have

$$\overline{\sigma}^0 \sigma^{\mu\nu} - \overline{\sigma}^\nu \sigma^{0\mu} = \frac{1}{2}\overline{\sigma}^0 \delta^{\mu\nu}, \tag{7.195}$$

and by the Nahm equation and the Jacobi identity,

$$\left[\frac{dT_\mu}{ds}, T_\mu\right] = \frac{i}{2}\epsilon_{\mu\nu\rho} \left[[T_\nu, T_\rho], T_\mu\right] = 0. \tag{7.196}$$

Therefore, the constraint is satisfied.

Note that this solution reduces to the trivial one ($\mathcal{M}_\gamma = 0$, $\lambda_\alpha = 0$) in the one monopole case where the T_μ are constant and commute with each other. This is as expected, because the supersymmetric modes should account for all of the one monopole zero modes.

We call the adjoint fermion zero modes given by this solution the dual supersymmetric modes, because they arise due to supersymmetry in the $U(k)$ gauge theory with fields T_m and $\mathcal{M}_\gamma$, which can be considered dual to our original $SU(2)$ theory containing fields v_m and λ_α.

This type of link, between solutions in different theories, was first introduced by Corrigan and Goddard[4] [75], in the context of both the ADHM and Nahm constructions. These ideas were reinterpreted and extended recently by Dorey and collaborators [23, 24], who showed that the ADHM data may be viewed as arising from the dimensional reduction of a ten dimensional gauge theory to six dimensions. Given the analogies between the Nahm data and a gauge field, seen in this section and also particularly in section 7.3.3, it seems probable that this link could also be made more precise for the Nahm construction.

7.6.3 Fundamental fermion zero modes

For completeness, we note that the solutions of the Dirac equation,

$$\overline{\sigma}^{m\dot{\alpha}\alpha} D_m \psi_\alpha = \overline{\sigma}^{m\dot{\alpha}\alpha} \left(\partial_m \psi_\alpha + v_m \psi_\alpha \right) = 0, \tag{7.197}$$

for fermions in the fundamental representation of the gauge group, are given by

$$(\psi^\alpha)^{\dot{\beta}} = \int ds\, \overline{U}^{i\dot{\beta}\alpha}(s) f_i{}^j(s,0) \mathcal{K}_j, \tag{7.198}$$

for any values of the Grassmannian parameters $\mathcal{K}_j$.

7.7 Two monopole solution

The two monopole solution was found using the Nahm construction by Panagopoulos [78]. We discuss some aspects of it here.

The first task, when finding a monopole configuration with any winding number k, is to solve for the Nahm data, $\{T_\mu\}$. We notice that the Nahm equation implies that the trace of any of the T matrices is constant,

$$i\frac{d}{ds}\mathrm{Tr}\, T_\mu = -\epsilon_{\mu\nu\rho}\mathrm{Tr}\, (T_\nu T_\rho) = 0. \tag{7.199}$$

Therefore, as in the one monopole case, we can identify the centre of the monopole configuration as

$$X_\mu = -\frac{1}{k}\mathrm{Tr}\, T_\mu, \tag{7.200}$$

[4] They used the term *reciprocity* "to avoid overworking the term duality".

and set this to zero for now; it can easily be restored by a translation. Also, differentiating the tensor

$$\Theta_{\mu\nu} = \mathrm{Tr}\,(T_\mu T_\nu) - \frac{1}{3}\delta_{\mu\nu}\mathrm{Tr}\,(T_\rho T_\rho), \tag{7.201}$$

with respect to s, the Nahm equation shows that this is constant as well, for example,

$$\frac{d}{ds}\mathrm{Tr}\,(T_1T_2) = i\mathrm{Tr}\,(T_2T_3T_2 - T_3T_2T_2 + T_3T_1T_1 - T_1T_1T_3) = 0, \tag{7.202}$$

$$\frac{d}{ds}\mathrm{Tr}\,(T_\rho T_\rho) = 6i\mathrm{Tr}\,(T_1T_2T_3 - T_3T_2T_1), \tag{7.203}$$

and

$$\frac{d}{ds}\mathrm{Tr}\,(T_1T_1) = 2i\mathrm{Tr}\,(T_1T_2T_3 - T_1T_3T_2) = \frac{1}{3}\frac{d}{ds}\mathrm{Tr}\,(T_\rho T_\rho). \tag{7.204}$$

This tensor is real and symmetric, and we can use a spatial rotation to transform it into a diagonal form. Then, $\mathrm{Tr}\,(T_\mu T_\nu) = 0$ for $\mu \neq \nu$.

This last condition, as well as tracelessness, will always be fulfilled in the two monopole case if

$$T_1 = f_1\frac{\tau_1}{2}, \quad T_2 = f_2\frac{\tau_2}{2}, \quad T_3 = f_3\frac{\tau_3}{2}. \tag{7.205}$$

Substituting these matrices into the Nahm equation, we find the following equations for the functions $\{f_\mu\}$,

$$\frac{df_1}{ds} = -f_2f_3, \quad \frac{df_2}{ds} = -f_3f_1, \quad \frac{df_3}{ds} = -f_1f_2. \tag{7.206}$$

The required solution is

$$f_1 = -\frac{2K}{u}\frac{(1-m)^{\frac{1}{2}}}{\mathrm{cn}\left(\frac{2Ks}{u}\right)}, \tag{7.207}$$

$$f_2 = -\frac{2K}{u}\frac{\mathrm{dn}\left(\frac{2Ks}{u}\right)}{\mathrm{cn}\left(\frac{2Ks}{u}\right)}, \tag{7.208}$$

$$f_3 = -\frac{2K}{u}\frac{(1-m)^{\frac{1}{2}}\mathrm{sn}\left(\frac{2Ks}{u}\right)}{\mathrm{cn}\left(\frac{2Ks}{u}\right)}, \tag{7.209}$$

where K is the elliptic integral of the first kind, and sn, cn and dn are Jacobi elliptic functions, and all of these depend on the parameter $m \in [0, 1]$. The definitions and some properties of these functions are given in appendix E. Using the information contained there, it is easy to show that this is a solution, and that the behaviour of all of the $\{f_\mu\}$ near $s = \pm\frac{u}{2}$ is $f_\mu \approx \frac{1}{s\mp\frac{u}{2}}$. Therefore, the residues of the T matrices at the endpoints of the s interval are indeed the generators of the two dimensional (fundamental) irreducible representation of $SU(2)$.

The $k = 2$ solution for $\{T_\mu\}$ above contains a new parameter, m, or equivalently K, which is a bosonic collective coordinate. However, we would expect eight bosonic collective coordinates for the most general two monopole solution, so what are the other seven? Three are the coordinates of the centre, X_μ, and a further one is the angle of an unbroken $U(1)$ gauge transformation, Ω; these are just like the one monopole collective coordinates. The remaining three parameters are the Euler angles of the rotation used to make $\Theta_{\mu\nu}$ diagonal, ϑ, φ, ψ. Brown, Panagopoulos and Prasad [79] showed that the solution above corresponds to having both poles on the x_2-axis. The collective coordinate m (or K) gives an indication of the separation of the individual monopoles; $m = 0$ and $K = \frac{\pi}{2}$ when they are overlapping, $m \to 1$ or $K \to \infty$ is the far-separated limit.

We know the bosonic part of the measure in terms of these bosonic collective coordinates, because, in a seminal work of mathematical physics [80], Atiyah and Hitchin determined the two monopole moduli space, or space of all two monopole solutions,

$$\mathfrak{M}_2 = \mathbb{R}^3 \times \frac{S^1 \times \mathfrak{M}_2^0}{\mathbb{Z}_2}. \tag{7.210}$$

The factor $\mathbb{R}^3 \times S^1$ is the contribution of the centre and $U(1)$ gauge orientation, and the $\mathbb{Z}_2$ factor appears because of the symmetry under exchange of the monopoles. The four dimensional relative moduli space $\mathfrak{M}_2^0$ is known as the Atiyah-Hitchin manifold, and the metric on this space is, using the notation of [81] (the relations in appendix E can be used to show that this is the same metric as written in [82]),

$$\begin{aligned} ds^2 = f^2 dK^2 &+ a^2(-\sin\psi\, d\vartheta + \cos\psi \sin\vartheta\, d\varphi)^2 \\ &+ b^2(\cos\psi\, d\vartheta + \sin\psi\sin\vartheta\, d\varphi)^2 \\ &+ c^2(d\psi + \cos\vartheta\, d\varphi)^2, \end{aligned} \tag{7.211}$$

where

$$a^2 = \frac{2K(K-E)(E-(1-m)K)}{E}, \tag{7.212}$$

$$b^2 = \frac{2KE(K-E)}{E-(1-m)K}, \tag{7.213}$$

$$c^2 = \frac{2KE(E-(1-m)K)}{K-E}, \tag{7.214}$$

$$f^2 = \frac{2E(K-E)}{K(E-(1-m)K)}. \tag{7.215}$$

The bosonic measure is given by the square root of the determinant of this metric (as well as the flat measure of $\mathbb{R}^3 \times S^1$), so up to factors,

$$d\mu^B_{2-\mathrm{mono}} = \int d^3X\, d\Omega\, dK\, d\vartheta\, d\varphi\, d\psi\, 4KE(K-E)\sin\vartheta. \tag{7.216}$$

Panagopoulos [78] indicated the method for finding the vector U in the Nahm construction, and also the gauge field, with the Nahm data as above. He did not write either out explicitly, but they are both complicated expressions involving Jacobi elliptic functions and depending partially implicitly on the spatial position through the solution, ζ, of the quartic equation,

$$\zeta^4\left(\frac{x_+^2}{4}+\frac{K^2m}{4u^2}\right)+\zeta^3\left(ix_+x_3\right)+\zeta^2\left(\frac{x_+x_-}{2}-x_3^2+\frac{K^2}{2u^2}(m-2)\right)$$
$$+\,\zeta\left(ix_-x_3\right)+\left(\frac{x_-^2}{4}+\frac{K^2m}{4u^2}\right)=0. \tag{7.217}$$

Such intricacy makes any attempt to directly calculate a correlation function, like that in equation (7.1), prohibitively difficult. Furthermore, there is no way to approximately calculate such a quantity, for example taking the dilute monopole gas limit, without neglecting important contributions. Therefore, it is not feasible to perform two monopole calculations to determine correlation functions.

Chapter 8

Conclusions

We have been investigating semiclassical calculations in $\mathcal{N} = 1$ supersymmetric gauge theories. Supersymmetry is of both phenomenological and fundamental interest, and it also implies a powerful renormalisation theorem, which allows us to calculate F-terms exactly.

The primary example that we have discussed is the evaluation of the gluino condensate in pure Yang-Mills theory. This is the order parameter for chiral symmetry breaking, and while every calculation shows that it is non-zero, its absolute magnitude has been the subject of controversy for many years. Using Yang-Mills instantons in $\mathbb{R}^4$, the most direct method to determine the gluino condensate is to calculate a higher multi-point function and invoke cluster decomposition. This is known as the SCI (strong coupling instanton) approach, because there is no control available over the coupling associated with the non-abelian gauge group. In contrast, there are several methods where the gluino condensate is calculated in a related theory, modified to ensure that the coupling is small, and then the appropriate limit is taken to return to the original theory. Every example of this WCI (weak coupling instanton) approach yields the same value of the gluino condensate, but there is a discrepancy between this value and the SCI result. Kovner and Shifman attempted to explain this disagreement using the drastic assumption of the existence of an extra vacuum state, with vanishing gluino condensate and unbroken chiral symmetry. Further instanton calculations have shown that this mechanism does not account for the difference.

The approach presented in this thesis provides a much more elegant understanding of the situation. It is motivated by the fact that there is no proof that instantons are the only relevant semiclassical configurations when the coupling is large, and by the long-standing idea that in this regime, instantons should be treated as composite objects, made of instanton partons. If there are neglected configurations and instantons do not give the full correlation function, then cluster decomposition is misapplied and the SCI result is invalid. The candidates for instanton partons on $\mathbb{R}^4$ are merons, which are not useful configurations for semiclassical calculations; for example, they have infinite action.

Instead of considering merons, the strategy is to modify space to $\mathbb{R}^3 \times S^1$, by imposing periodicity along one direction, on all the fields (both bosonic and fermionic, to preserve supersymmetry). Then, the partons become well-behaved, finite action configurations. Additionally, we gain control over the size of the coupling, because the component of the gauge field around the periodic direction can gain a non-zero VEV, and act as a Higgs field to spontaneously break the gauge group to its maximal abelian subgroup. The theory is then in a Coulomb phase, and the coupling will be small if the VEV parameter is arranged to be much larger than the dynamically generated scale. A semiclassical calculation of the low energy effective action of the theory on $\mathbb{R}^3 \times S^1$, discussed in chapter 5, shows that in the true quantum vacuum the VEV parameter is inversely proportional to the radius of the circle. We may therefore choose the radius to be small to guarantee weak coupling, but according to supersymmetry all results should be holomorphic in the VEV, and therefore also in the radius, so we can analytically continue to large values of the radius and recover the $\mathbb{R}^4$ answer. Working on the cylinder therefore has similarities to the WCI methods, but it also allows us to visualise the configurations that are missed in the SCI approach. The connection between the VEV and the radius is only found for $\mathbb{R}^3 \times S^1$, so other compact spaces such as T^4 are not appropriate for attempts to calculate quantities in $\mathbb{R}^4$.

The semiclassical configurations on $\mathbb{R}^3 \times S^1$ can be classified as the combinations of $n+1$ types of monopoles (where n is the rank of the gauge group), distinguished by magnetic charges as well as the instanton number. The different fundamental monopoles are the n conventional BPS monopoles, and one KK or affine monopole, which is only present because the space is a four dimensional cylinder. More precisely, it appears due to the existence of non-periodic gauge transformations, which nevertheless leave the fields periodic, but that are not included in the group of periodic gauge transformations under which the functional integral measure is invariant. One specific combination of these monopoles is a caloron (with non-trivial VEV or Wilson loop), the analogous configuration to an $\mathbb{R}^4$ instanton, so the fundamental monopoles correspond to instanton partons. Any other combination of the same number of monopoles represents a configuration that is ignored by the SCI calculation.

As we mentioned above, we have determined the low energy dynamics of the theory on $\mathbb{R}^3 \times S^1$, in the form of the superpotential for a chiral superfield containing the Wilson loop and a scalar magnetic photon. The superpotential is a twisted affine Toda potential, as shown in chapter 6, and as predicted by Katz and Vafa. Expanding the superpotential we can see that the magnetic photon is massive, which by the dual Meissner effect shows that the original electric charges will be confined. Instanton partons were first studied because it was thought that they might be responsible for confinement, and here we find that this is exactly the case for $\mathcal{N} = 1$ supersymmetric theories.

Each of the fundamental monopoles has two adjoint fermion zero modes, so the gluino condensate may be calculated by summing their direct contributions. It was determined for gauge group $SU(2)$ in chapter 5, and for any gauge group in chapter 6 (see also [1] and [3]), with the results summarised in table 6.2. The magnitude agrees with those given by previous WCI calculations for all classical groups, whereas the values for the exceptional groups have not been predicted before. The phase labels the c_2 physically equivalent vacua of the theory (where c_2 is the dual Coxeter number of the gauge group), a total that is fully consistent with Witten's index.

In chapter 6 we also discussed a useful description of a supersymmetric gauge theory including matter, namely the ADS superpotential that describes the low energy dynamics of the classically massless matter fields, in a theory with gauge group $SU(N)$ and $N_f < N$ flavours of matter. It was originally calculated in the case $N_f = N - 1$ using instantons, and then inferred for all other values by renormalisation group matching. However, because of its connection with gluino condensation, for $N_f < N - 1$ the ADS superpotential has been shown to be generated directly by monopoles [2].

For gauge group $SU(2)$, the configurations that are not included in the SCI approach are the $\mathbb{R}^4$ analogues of the combinations of two BPS monopoles, or two KK monopoles. In chapter 7 we reported on the progress made in an attempt to perform a semiclassical calculation with these configurations, to confirm that they contribute the difference between the WCI and SCI results for the gluino condensate. The approach involved attempting to understand the two monopole solution through the Nahm construction, which we reviewed, using the one monopole case as an example. Some new results were obtained, in the form of an identity for the Green's function that appears in the Nahm construction, and two adjoint fermion zero modes possessed by all monopoles with winding number greater than one. Unfortunately, however, two monopole calculations eventually proved too complicated to be workable. In principle, instantons and monopoles are the only semiclassical configurations necessary to find any F-term in four dimensional supersymmetric gauge theories, but because of technical limitations there are only a few results that can be explicitly derived using monopoles.

Appendix A

Conventions

We work in four dimensional space with a Euclidean metric $(++++)$, unless otherwise specified. Latin letters $m, n, \ldots$ run over $0,1,2,3$; Greek $\mu, \nu, \ldots$ over $1,2,3$. In addition there are undotted and dotted Weyl indices $\alpha, \beta, \dot{\alpha}, \dot{\beta}, \ldots$ which take the values 1 and 2. Writing the Pauli matrices as

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.1}$$

we can define Dirac-Weyl matrices

$$\sigma^m{}_{\alpha\dot{\alpha}} = (1, i\vec{\tau})_{\alpha\dot{\alpha}}, \tag{A.2}$$

$$\overline{\sigma}^{m\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\sigma^m{}_{\beta\dot{\beta}} = (1, -i\vec{\tau})^{\dot{\alpha}\alpha}, \tag{A.3}$$

$$\sigma^{mn} = \frac{1}{4}(\sigma^m\overline{\sigma}^n - \sigma^n\overline{\sigma}^m), \tag{A.4}$$

$$= \begin{cases} -\frac{i}{2}\tau_\nu & m = 0, n = \nu, \\ \frac{i}{2}\epsilon^{\mu\nu\rho}\tau_\rho & m = \mu, n = \nu, \end{cases} \tag{A.5}$$

$$= \frac{1}{2}\epsilon^{mnpq}\sigma_{pq}, \tag{A.6}$$

$$\overline{\sigma}^{mn} = \frac{1}{4}(\overline{\sigma}^m\sigma^n - \overline{\sigma}^n\sigma^m), \tag{A.7}$$

$$= \begin{cases} \frac{i}{2}\tau_\nu & m = 0, n = \nu, \\ \frac{i}{2}\epsilon^{\mu\nu\rho}\tau_\rho & m = \mu, n = \nu, \end{cases} \tag{A.8}$$

$$= -\frac{1}{2}\epsilon^{mnpq}\overline{\sigma}_{pq}, \tag{A.9}$$

where $\epsilon^{12} = -\epsilon_{12} = +1$, $\epsilon^{123} = \epsilon_{123} = +1$ and $\epsilon^{0123} = \epsilon_{0123} = -1$. With these conventions, the Clifford algebra is

$$\overline{\sigma}^{m\dot{\alpha}\alpha}\sigma^{n}{}_{\alpha\dot{\beta}} + \overline{\sigma}^{n\dot{\alpha}\alpha}\sigma^{m}{}_{\alpha\dot{\beta}} = +2\delta^{mn}\delta^{\dot{\alpha}}{}_{\dot{\beta}}, \tag{A.10}$$

and both $\{\sigma^{mn}\}$ and $\{\overline{\sigma}^{mn}\}$ obey the $SO(4)$ algebra,

$$[\sigma^{mn}, \sigma^{pq}] = \delta^{np}\sigma^{mq} + \delta^{mq}\sigma^{np} + \delta^{qn}\sigma^{pm} + \delta^{pm}\sigma^{qn}, \tag{A.11}$$

$$[\overline{\sigma}^{mn}, \overline{\sigma}^{pq}] = \delta^{np}\overline{\sigma}^{mq} + \delta^{mq}\overline{\sigma}^{np} + \delta^{qn}\overline{\sigma}^{pm} + \delta^{pm}\overline{\sigma}^{qn}. \tag{A.12}$$

In fact, the $\{\sigma^{mn}\}$ and $\{\overline{\sigma}^{mn}\}$ are the generators of the $(2,1)$ and $(1,2)$ representations of $SO(4) \approx SU(2)_L \times SU(2)_R/\mathbb{Z}_2$, respectively. The following relations are also useful,

$$\sigma^{m}{}_{\alpha\dot{\alpha}}\overline{\sigma}_{m}{}^{\dot{\beta}\beta} = +2\delta^{\beta}{}_{\alpha}\delta^{\dot{\beta}}{}_{\dot{\alpha}}, \tag{A.13}$$

$$\sigma^{mn}{}_{\alpha}{}^{\beta}\sigma^{kl}{}_{\beta}{}^{\alpha} = -\frac{1}{2}\left(\delta^{mk}\delta^{nl} - \delta^{ml}\delta^{nk}\right) - \frac{1}{2}\epsilon^{mnkl}, \tag{A.14}$$

$$\sigma^{m}\overline{\sigma}^{n}\sigma^{p} - \sigma^{p}\overline{\sigma}^{n}\sigma^{m} = -2\epsilon^{mnpq}\sigma_{q}, \tag{A.15}$$

$$\overline{\sigma}^{m}\sigma^{n}\overline{\sigma}^{p} - \overline{\sigma}^{p}\sigma^{n}\overline{\sigma}^{m} = +2\epsilon^{mnpq}\overline{\sigma}_{q}, \tag{A.16}$$

$$\sigma^{m}\overline{\sigma}^{n}\sigma^{p} + \sigma^{p}\overline{\sigma}^{n}\sigma^{m} = +2\left(\delta^{mn}\sigma^{p} + \delta^{pn}\sigma^{m} - \delta^{mp}\sigma^{n}\right), \tag{A.17}$$

$$\overline{\sigma}^{m}\sigma^{n}\overline{\sigma}^{p} + \overline{\sigma}^{p}\sigma^{n}\overline{\sigma}^{m} = +2\left(\delta^{mn}\overline{\sigma}^{p} + \delta^{pn}\overline{\sigma}^{m} - \delta^{mp}\overline{\sigma}^{n}\right). \tag{A.18}$$

Appendix B

Simple Lie algebras

In this section we shall briefly review some of the theory of simple Lie algebras in order to state our choices of conventions and normalisations, and to define important terms. There are more comprehensive introductions in many books, see for example [83]. Some other useful information is given in [84] and the appendix of [68], and informal descriptions of some rarely covered topics can be found in weeks 64 and 90 (amongst others) of [85].

Let $\{H^i\}$ be a maximal set of simultaneously diagonalisable, mutually commuting generators, $[H^i, H^j] = 0$. The indices i, j run from 1 to n, the *rank* of the Lie algebra. The span of $\{H^i\}$ is h, the Cartan subalgebra. For the remainder of the generators, we choose such combinations that they are eigenvectors E_α of the operators $\mathrm{ad}(H^i)$,

$$\mathrm{ad}(H^i)E_\alpha \equiv [H^i, E_\alpha] = \alpha^i E_\alpha. \tag{B.1}$$

Now we shift focus and consider the eigenvalue α^i to be the value of a linear functional α acting on the basis vector H^i,

$$[H^i, E_\alpha] = \alpha(H^i)E_\alpha. \tag{B.2}$$

These functionals are called *roots* and are elements of h^*, the root space. Obviously $\dim(h^*) = \dim(h) = n$, and there is one root α for every generator E_α. It turns out to be easier to consider the structure of Lie algebras by using the roots and the Cartan subalgebra generators, rather than the set of all the generators.

We can define an ordering on the roots by expanding them in any given basis $\{\beta_{(i)}\}$,

$$\alpha = \sum_{i=1}^{n} b^i \beta_{(i)}, \tag{B.3}$$

then α is a positive root if the first non-zero component b^i is positive, and similarly for negative roots. Clearly which roots are positive and negative depends on our arbitrary choice of basis, but the alternatives can be easily related via group transformations, they are not substantially different. We can then arrive at a more convenient and natural basis for h^* by finding the *simple roots.* A root is simple if it is a positive root which cannot be written as the sum of two other positive roots. We shall denote the simple roots as $\{\alpha_{(i)}, i = 1 \ldots n\}$, and then we can write any root as

$$\alpha = \sum_{i=1}^{n} a^i \alpha_{(i)}, \tag{B.4}$$

where the coefficients a^i are positive integers or zero for positive roots, and negative integers or zero for negative roots. Another distinguished root is the *highest root* θ, which has the maximum value of $\sum_{i=1}^{n} a^i$.

The overall length scale of the roots is clearly linked to the normalisation of the Cartan subalgebra generators, and we shall follow the standard algebraists' choice that

$$\mathrm{Tr}\,(H^i H^j) = \delta^{ij}. \tag{B.5}$$

In some simple Lie algebras (the simply laced ones), all roots have the same length squared, L. For non-simply laced algebras there are two classes of roots, long roots and short roots. We define the length squared of the long roots to be L, for example the highest root is always a long root, so

$$|\theta|^2 \equiv \sum_{i=1}^{n} \theta^i \theta^i = L. \tag{B.6}$$

The short roots may have length squared $\frac{L}{2}$ or $\frac{L}{3}$. To complete our discussion of normalisations, the eigenvectors E_α are scaled such that the convenient relation

$$[E_\alpha, E_{-\alpha}] = \frac{L}{|\alpha|^2} \sum_{i=1}^{n} \alpha^i H^i \equiv \frac{L}{|\alpha|^2} \alpha \cdot H, \tag{B.7}$$

holds.

We can form duals of the roots, or *coroots* by inverting their lengths, so

$$\alpha^* \equiv \frac{L}{|\alpha|^2}\alpha. \tag{B.8}$$

Note that the coroots are still elements of h^*, and that for long roots (or all roots of simply laced algebras), $\alpha^* = \alpha$. The simple coroots and highest coroot are defined to be the duals of the simple roots and highest root respectively[1]. The highest coroot can be expanded in terms of the simple coroots,

$$\theta^* = \sum_{i=1}^{n} m^i \alpha^*_{(i)}, \tag{B.9}$$

where the coefficients m^i are called comarks and

$$c_2 \equiv 1 + \sum_{i=1}^{n} m^i \tag{B.10}$$

is the *dual Coxeter number* (the marks and Coxeter number are similarly defined but will not be needed here). The fundamental weights and coweights are defined through

$$\omega^{(i)} \cdot \alpha^*_{(j)} = \delta^i{}_j, \tag{B.11}$$

and

$$\omega^{(i)}_* \cdot \alpha_{(j)} = \delta^i{}_j, \tag{B.12}$$

respectively, and the Weyl vector is

$$\rho = \sum_{i=1}^{n} \omega^{(i)}. \tag{B.13}$$

The simple Lie algebras are classified as shown in tables B.1 and B.2. The simply laced algebras are A_n, D_n and $E_{6,7,8}$. Some relevant data are listed for all the simple Lie algebras, including the comarks. A bar over the comark indicates that it is associated with a short root. In B_n, C_n and F_4, the short roots have length squared $\frac{L}{2}$, in G_2 they have length squared $\frac{L}{3}$. Remember that there are always n comarks, where n is the rank.

[1]Thus, for example, if we write $\alpha^* = \sum_{i=1}^{n} a^{*i}\alpha^*_{(i)}$ then the highest coroot is not necessarily the coroot with the greatest value of $\sum_{i=1}^{n} a^{*i}$.

Algebra	Compact group	Rank	Dimension	Dual Coxeter no.	Comarks
A_n	$SU(n+1)$	$n \geqslant 1$	$n(n+2)$	$n+1$	$1, \ldots, 1$
B_n	$SO(2n+1)$	$n \geqslant 2$	$n(2n+1)$	$2n-1$	$1, 2, \ldots, 2, \bar{1}$
C_n	$USp(n)$	$n \geqslant 3$	$n(2n+1)$	$n+1$	$\bar{1}, \ldots, \bar{1}, 1$
D_n	$SO(2n)$	$n \geqslant 4$	$n(2n-1)$	$2n-2$	$1, 2, \ldots, 2, 1, 1$

Notes:

- We use the notation $USp(n)$ to refer to the compact group associated with the algebra C_n, that is, $USp(n)$ is the group of $2n \times 2n$ special unitary matrices which preserve the symplectic form. This is often alternatively referred to as $USp(2n), Sp(n)$ or $Sp(2n)$. We prefer to keep $Sp(n)$ for the group of real $2n \times 2n$ matrices which preserve the symplectic form.

- $SO(2) \approx U(1)$ and $SO(4) \approx SU(2) \times SU(2)/\mathbb{Z}_2$ are not simple. The remaining restrictions on the rank come from wanting to consider only distinct cases, for example $SO(3) \approx SU(2)/\mathbb{Z}_2$ hence $n \geqslant 2$ for B_2. Also recall $USp(1) \approx SU(2)$, $SO(5) \approx USp(2)/\mathbb{Z}_2$, $SO(6) \approx SU(4)/\mathbb{Z}_2$.

Table B.1: The classical simple Lie algebras of rank n.

Algebra	Rank	Dimension	Dual Coxeter no.	Comarks
E_6	6	78	12	$1, 2, 3, 2, 1, 2$
E_7	7	133	18	$2, 3, 4, 3, 2, 1, 2$
E_8	8	248	30	$2, 3, 4, 5, 6, 4, 2, 3$
F_4	4	52	9	$2, 3, \bar{2}, \bar{1}$
G_2	2	14	4	$2, \bar{1}$

Table B.2: The exceptional simple Lie algebras.

Appendix C

Monopole measures

C.1 The $SU(2)$ one monopole measure

This section follows the appendix C of [48] closely, in order to find the one monopole measure with gauge group $SU(2)$. In the next section we shall generalise the results to monopoles in any simple Lie group.

To determine the measure, we need to know the collective coordinates and calculate the Jacobian factors, like that given in equation (2.30), which we repeat here for convenience,

$$\mathcal{J}_B = \left(\frac{1}{\sqrt{2\pi}}\right)^{n_B} \det \left\langle \frac{\partial}{\partial \tau_j}\phi_{\text{class}}, \frac{\partial}{\partial \tau_k}\phi_{\text{class}} \right\rangle^{\frac{1}{2}}, \tag{C.1}$$

Considering the bosonic part first, the collective coordinates are X_μ, the position of the monopole, and Ω, the $U(1)$ angle, as discussed in chapter 4. The bosonic Jacobian factor given above is written for a scalar field, in which case the unnormalised zero modes are the derivatives of the classical solution with respect to the collective coordinates. We can write similar expressions for a gauge field,

$$\mathcal{J}_B = \left(\frac{1}{\sqrt{2\pi}}\right)^{n_B} \det \left\langle Z^{[j]m}, Z^{[k]}_m \right\rangle^{\frac{1}{2}}, \tag{C.2}$$

with

$$\langle f, g \rangle = \frac{1}{g^2} \int d^4x \, \mathrm{Tr}\, (f(x)g(x)), \tag{C.3}$$

and

$$Z_m^{[j]} = \frac{\partial}{\partial \tau_j} v_m^{\text{class}}. \tag{C.4}$$

However, this is not the only possible way of writing the zero modes $\{Z_m^{[j]}\}$, as they inherit some gauge dependence from v_m. If $v_m \mapsto e^{i\Lambda} v_m e^{-i\Lambda} + i e^{i\Lambda} \partial_m e^{-i\Lambda}$, then because Z_m is a first order shift away from v_m^{class}, the correct transformation law for the zero modes is the infinitesimal version, $Z_m \mapsto Z_m + D_m^{\text{class}} \Lambda$. Therefore, the general form of the bosonic zero modes is

$$Z_m^{[j]} = \frac{\partial}{\partial \tau_j} v_m^{\text{class}} + D_m^{\text{class}} \Lambda^{[j]}. \tag{C.5}$$

As discussed in section 2.1.4, it is necessary to fix the gauge, which means the functions $\{\Lambda^{[j]}\}$ should be chosen to ensure that the gauge fixing condition is satisfied. In this thesis we work in the covariant background gauge where

$$D_m^{\text{class}} (\delta v^m) = 0, \tag{C.6}$$

which implies the zero modes must also obey

$$D_m^{\text{class}} Z^{[j]m} = 0. \tag{C.7}$$

In the case of the translational zero modes,

$$Z_{[\mu]m} = \frac{\partial}{\partial X^\mu} v_m^{\text{class}} + D_m^{\text{class}} \Lambda_{[\mu]}, \tag{C.8}$$

the classical solution is a function of $x - X$, so we can also write

$$Z_{[\mu]m} = -\frac{\partial}{\partial x^\mu} v_m^{\text{class}} + D_m^{\text{class}} \Lambda_{[\mu]}. \tag{C.9}$$

Then if we set $\Lambda_{[\mu]} = v_\mu^{\text{class}}$, $Z_{[\mu]m} = v_{m\mu}^{\text{class}}$, which obviously fulfils equation (C.7) as v_m^{class} is a solution of the classical equations of motion. This leads to

$$\left\langle Z_{[\mu]}^m, Z_{[\nu]m} \right\rangle = S \delta_{\mu\nu}, \tag{C.10}$$

where S is the action of the monopole configuration.

We can follow a similar calculation for the gauge orientation zero mode,

$$Z_{[4]m} = \frac{\partial}{\partial \Omega} v_m^{\text{class}} + D_m^{\text{class}} \Lambda_{[4]}. \tag{C.11}$$

If we work with the BPS monopole in the singular gauge, where

$$v_0^{\text{class}} = \Phi\frac{\tau_3}{2} \overset{|x|\to\infty}{\longrightarrow} -u\frac{\tau_3}{2}, \tag{C.12}$$

so that the zero mode corresponds to the global gauge transformations $\{e^{i\Omega\frac{\tau_3}{2}}\}$, then

$$Z_{[4]m} = \frac{\partial}{\partial\Omega}\left(e^{i\Omega\frac{\tau_3}{2}}\hat{v}_m^{\text{class}}e^{-i\Omega\frac{\tau_3}{2}}\right) + D_m^{\text{class}}\Lambda_{[4]}, \tag{C.13}$$

where $\hat{v}_m^{\text{class}}$ is an Ω-independent reference configuration. Furthermore, we should choose $\Lambda_{[4]} = \frac{1}{u}\left(\Phi - \frac{\tau_3}{2}\right)$, then $Z_{[4]m} = \frac{1}{u}v_{m0}^{\text{class}}$, and

$$\left\langle Z_{[\mu]}^m, Z_{[4]m}\right\rangle = 0, \quad \left\langle Z_{[4]}^m, Z_{[4]m}\right\rangle = \frac{1}{u^2}S. \tag{C.14}$$

We can now put these results together to find

$$\mathcal{J}_B^{\text{BPS}} = \left(\frac{1}{\sqrt{2\pi}}\right)^{n_B}\det\left\langle Z^{[j]m}, Z_m^{[k]}\right\rangle^{\frac{1}{2}} = \frac{S^2}{4\pi^2 u}. \tag{C.15}$$

In order to find the fermionic part we simply need the normalisation of the supersymmetric zero modes, $\lambda^{\text{class}} = \sigma^{mn}\xi v_{mn}^{\text{class}}$. Using equation (A.14) we can evaluate[1]

$$\frac{1}{g^2}\int d^4x\int d^2\xi\, \operatorname{Tr}\lambda^{\text{class}}\lambda^{\text{class}} = 2S, \tag{C.16}$$

therefore, $\mathcal{J}_F = \frac{1}{2S}$.

The action of the BPS monopole is $\frac{8\pi^2 uR}{g^2}$, so bringing everything together we find the one monopole measure[2] in $SU(2)$,

$$\int d\mu_{1-\text{mono}} = \frac{\mu^3 R}{g^2}e^{-S}\int d^3X\int_0^{2\pi} d\Omega\int d^2\xi. \tag{C.17}$$

The KK monopole is gauge equivalent to a BPS monopole with a modified VEV, so the above equation is valid for either monopole as long as the correct action is used.

C.2 The one monopole measure for a general gauge group

Recall that a regular gauge monopole solution, in any gauge group, is given by

$$v_\mu^{\text{class}} = w_\mu^c J_c, \tag{C.18}$$

$$v_0^{\text{class}} = \Phi^c J_c - \left(V - \frac{1}{L}(\alpha\cdot V)\alpha^*\right)\cdot H, \tag{C.19}$$

[1] Note the obvious definition $\int d^2\xi\,\xi\xi = 1$.

[2] Note that the range of the Ω integration is 0 to 2π not 0 to 4π, as might be expected for an $SU(2)$ transformation, since v_m is in the adjoint representation and so a transformation by 2π is equivalent to the identity.

where

$$J_1 = \frac{1}{\sqrt{2L}}(E_\alpha + E_{-\alpha}), \quad J_2 = \frac{1}{\sqrt{2L}i}(E_\alpha - E_{-\alpha}), \quad J_3 = \frac{1}{L}\alpha^* \cdot H, \tag{C.20}$$

and if the monopole is located at the origin and no unbroken $U(1)$ transformation is applied, then

$$w^c_\mu = \epsilon_{\mu\nu c}\frac{x^\nu}{|x|^2}\left(1 - \frac{u|x|}{\sinh u|x|}\right), \tag{C.21}$$

$$\Phi^c = -\frac{x^c}{|x|^2}\left(u|x|\coth u|x| - 1\right), \tag{C.22}$$

with $u = \alpha \cdot V$.

One monopoles in any gauge group have the same zero mode structure as those in $SU(2)$. The bosonic zero modes are associated with the position and gauge orientation of the monopole, and the fermionic zero modes are the supersymmetric modes. The Jacobian factors for the translational bosonic zero modes and the fermionic zero modes are the same as in $SU(2)$, $\left(\frac{S}{2\pi}\right)^{\frac{3}{2}}$ and $\frac{1}{2S}$ respectively. This follows since the methods used to calculate these constants in the previous section did not refer to any special features of $SU(2)$ and can be seen to generalise immediately.

Calculating the Jacobian factor for the gauge orientation mode requires more attention. If the monopole is transformed into singular gauge, then the global gauge transformations corresponding to this zero mode are $\{e^{i\Omega J_3}\}$, therefore

$$v^{\text{class}}_m = e^{i\Omega J_3}\hat{v}^{\text{class}}_m e^{-i\Omega J_3}, \tag{C.23}$$

and the zero mode is

$$Z_{[4]m} = \frac{\partial}{\partial\Omega}v^{\text{class}}_m + D^{\text{class}}_m\Lambda_{[4]}. \tag{C.24}$$

In this case the correct choice of $\Lambda_{[4]}$ is

$$\Lambda_{[4]} = -\frac{1}{\alpha \cdot V}\Phi^c J_c - \frac{1}{L}\alpha^* \cdot H \tag{C.25}$$

(note that

$$\Phi^c J_c \to -\frac{1}{L}(\alpha \cdot V)\alpha^* \cdot H, \tag{C.26}$$

in singular gauge, as $|x| \to \infty$). Then,

$$Z_{[4]m} = -\frac{1}{\alpha \cdot V}v^{\text{class}}_{m0}. \tag{C.27}$$

It follows that the remaining Jacobian factor is $\frac{1}{\alpha \cdot V}\left(\frac{S}{2\pi}\right)^{\frac{1}{2}}$. Gathering all factors together, we find that the measure is

$$\int d\mu_{1-\text{mono}} = \frac{\mu^3 R}{g^2} e^{-S} \left(\frac{L}{|\alpha|^2}\right) \int d^3X \int d\Omega \int d^2\xi. \tag{C.28}$$

where $S = \frac{8\pi^2 R}{g^2}\alpha^* \cdot V$ is the action of the monopole configuration.

Appendix D

One monopole solution from the Nahm construction

See chapter 7 for the methods by which these results were calculated.

D.1 The gauge field components v_2 and v_3

$$v_2 = \begin{pmatrix} \frac{-ix_1}{2|x|(|x|-x_3)} & AB\left(-ix_1 + \frac{x_2x_3}{|x|}\right)u \\ AB\left(-ix_1 - \frac{x_2x_3}{|x|}\right)u & \frac{-ix_1}{2|x|(|x|+x_3)} \end{pmatrix}. \tag{D.1}$$

$$\hat{v}_2 = -\frac{i}{2|x|^2}\left(1 - \frac{u|x|}{\sinh u|x|}\right)\begin{pmatrix} -x_1 & x_3 \\ x_3 & x_1 \end{pmatrix}. \tag{D.2}$$

$$v_2'' = \check{v}_2 = \begin{pmatrix} \frac{ix_1}{2|x|(|x|+x_3)} & \frac{(ix_1|x| - x_2x_3)\,ux_-}{2|x|(|x|^2 - x_3^2)\sinh u|x|} \\ \frac{(ix_1|x| + x_2x_3)\,ux_+}{2|x|(|x|^2 - x_3^2)\sinh u|x|} & \frac{-ix_1}{2|x|(|x|+x_3)} \end{pmatrix}. \tag{D.3}$$

$$v_3 = \begin{pmatrix} 0 & -AB\frac{(|x|^2-x_3^2)}{|x|}u \\ AB\frac{(|x|^2-x_3^2)}{|x|}u & 0 \end{pmatrix}. \tag{D.4}$$

$$\hat{v}_3 = \frac{1}{2|x|^2}\left(1 - \frac{u|x|}{\sinh u|x|}\right)\begin{pmatrix} 0 & -x_- \\ x_+ & 0 \end{pmatrix}. \tag{D.5}$$

$$v_3'' = \check{v}_3 = \frac{u}{2|x|\sinh u|x|}\begin{pmatrix} 0 & x_- \\ -x_+ & 0 \end{pmatrix}. \tag{D.6}$$

D.2 The field strength component v_{23}

$$(v_{23})^1{}_1 = \frac{ix_1}{2|x|\sinh^2(u|x|)}\left\{\frac{1}{|x|^2}\sinh^2(u|x|) - u^2\right\}. \tag{D.7}$$

$$(v_{23})^1{}_2 = \frac{iu}{2|x|^2\sinh^2 u|x|}\frac{(i|x|x_2 + x_3x_1)}{(|x|^2 - x_3^2)^{\frac{1}{2}}}(\sinh(u|x|) - u|x|\cosh(u|x|)). \tag{D.8}$$

$$\begin{aligned}\hat{v}_{23} = &\frac{1}{2i|x|^2}\left[\tau_1 - \frac{x_1}{|x|^2}\begin{pmatrix} x_3 & x_- \\ x_+ & -x_3 \end{pmatrix}\right]\frac{u|x|}{\sinh(u|x|)}(1 - u|x|\coth(u|x|)) \\ &+ \frac{x_1}{2i|x|^4}\begin{pmatrix} x_3 & x_- \\ x_+ & -x_3 \end{pmatrix}\left(\frac{u^2|x|^2}{\sinh^2(u|x|)} - 1\right).\end{aligned} \tag{D.9}$$

$$(v_{23}'')^1{}_1 = (\check{v}_{23})^1{}_1 = -(\check{v}_{23})^2{}_2 = \frac{ix_1}{2|x|\sinh^2(u|x|)}\left\{\frac{1}{|x|^2}\sinh^2(u|x|) - u^2\right\}. \tag{D.10}$$

$$(v_{23}'')^1{}_2 = (\check{v}_{23})^1{}_2 = -\frac{iux_-}{2|x|^2\sinh^2 u|x|}\frac{(i|x|x_2 + x_3x_1)}{(|x|^2 - x_3^2)}(\sinh(u|x|) - u|x|\cosh(u|x|)). \tag{D.11}$$

$$(v_{23}'')^2{}_1 = (\check{v}_{23})^2{}_1 = +\frac{iux_+}{2|x|^2\sinh^2 u|x|}\frac{(i|x|x_2 - x_3x_1)}{(|x|^2 - x_3^2)}(\sinh(u|x|) - u|x|\cosh(u|x|)). \tag{D.12}$$

D.3 The field strength component v_{31}

$$(v_{31})^1{}_1 = \frac{ix_2}{2|x|\sinh^2(u|x|)}\left\{\frac{1}{|x|^2}\sinh^2(u|x|) - u^2\right\}. \tag{D.13}$$

$$(v_{31})^1{}_2 = \frac{iu}{2|x|^2\sinh^2 u|x|}\frac{(-i|x|x_1 + x_2x_3)}{(|x|^2 - x_3^2)^{\frac{1}{2}}}(\sinh(u|x|) - u|x|\cosh(u|x|)). \tag{D.14}$$

$$\hat{v}_{31} = \frac{1}{2i|x|^2}\left[\tau_2 - \frac{x_2}{|x|^2}\begin{pmatrix} x_3 & x_- \\ x_+ & -x_3 \end{pmatrix}\right]\frac{u|x|}{\sinh(u|x|)}\left(1 - u|x|\coth(u|x|)\right)$$

$$+\frac{x_2}{2i|x|^4}\begin{pmatrix} x_3 & x_- \\ x_+ & -x_3 \end{pmatrix}\left(\frac{u^2|x|^2}{\sinh^2(u|x|)} - 1\right). \tag{D.15}$$

$$(v''_{31})^1{}_1 = (\check{v}_{31})^1{}_1 = -(\check{v}_{31})^2{}_2 = \frac{ix_2}{2|x|\sinh^2(u|x|)}\left\{\frac{1}{|x|^2}\sinh^2(u|x|) - u^2\right\}. \tag{D.16}$$

$$(v''_{31})^1{}_2 = (\check{v}_{31})^1{}_2 = +\frac{iux_-}{2|x|^2\sinh^2 u|x|}\frac{(i|x|x_1 - x_2x_3)}{(|x|^2 - x_3^2)}\left(\sinh(u|x|) - u|x|\cosh(u|x|)\right). \tag{D.17}$$

$$(v''_{31})^2{}_1 = (\check{v}_{31})^2{}_1 = -\frac{iux_+}{2|x|^2\sinh^2 u|x|}\frac{(i|x|x_1 + x_2x_3)}{(|x|^2 - x_3^2)}\left(\sinh(u|x|) - u|x|\cosh(u|x|)\right). \tag{D.18}$$

Appendix E

Jacobi elliptic functions and elliptic integrals

In this appendix we give the definitions and some properties of the Jacobi elliptic functions and the related elliptic integrals, which appear in two monopole solutions. Further details can be found in [86].

The elliptic integrals of the first and second kinds are defined using the *parameter* or *modulus*, m, a real number in the interval $[0,1]$, as

$$F(\varphi, m) = \int_0^{\varphi} \frac{d\theta}{\left(1 - m\sin^2\theta\right)^{\frac{1}{2}}}, \tag{E.1}$$

$$E(\varphi, m) = \int_0^{\varphi} d\theta \, \left(1 - m\sin^2\theta\right)^{\frac{1}{2}}. \tag{E.2}$$

The parameter is also commonly denoted as k^2 or $\sin^2\alpha$.

The Jacobi elliptic functions are defined with the *argument*, u, and the *amplitude*, φ, which are related to each other through

$$u(\varphi) = F(\varphi, m), \tag{E.3}$$

and the inverse of this function, which is written as $\varphi = \mathrm{am}(u)$. Then, the Jacobi elliptic functions sn, cn and dn are defined as

$$\mathrm{sn}\,(u, m) = \sin\varphi, \tag{E.4}$$

$$\mathrm{cn}\,(u, m) = \cos\varphi, \tag{E.5}$$

$$\mathrm{dn}\,(u, m) = \left(1 - m\sin^2\varphi\right)^{\frac{1}{2}}. \tag{E.6}$$

We shall normally let the dependence on the parameter m be implicit. We can already tell from the definitions that

$$\mathrm{sn}^2 u + \mathrm{cn}^2 u = 1, \tag{E.7}$$

$$m\,\mathrm{sn}^2 u + \mathrm{dn}^2 u = 1. \tag{E.8}$$

The complete elliptic integral of the first kind is

$$K = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\left(1 - m\sin^2\theta\right)^{\frac{1}{2}}} = u\left(\frac{\pi}{2}\right). \tag{E.9}$$

We may easily evaluate the Jacobi elliptic functions at the points $u = 0$ and $u = K$,

$$\mathrm{sn}\,0 = 0, \qquad \mathrm{sn}\,K = 1, \tag{E.10}$$

$$\mathrm{cn}\,0 = 1, \qquad \mathrm{cn}\,K = 0, \tag{E.11}$$

$$\mathrm{dn}\,0 = 1, \quad \mathrm{dn}\,K = (1-m)^{\frac{1}{2}}. \tag{E.12}$$

If we note that $\frac{du}{d\varphi} = \frac{1}{\mathrm{dn}\,u}$, so that $\frac{d\varphi}{du} = \mathrm{dn}\,u$, we can differentiate the Jacobi elliptic functions,

$$\frac{d}{du}\,\mathrm{sn}\,u = \mathrm{cn}\,u\;\mathrm{dn}\,u, \tag{E.13}$$

$$\frac{d}{du}\,\mathrm{cn}\,u = -\mathrm{dn}\,u\;\mathrm{sn}\,u, \tag{E.14}$$

$$\frac{d}{du}\,\mathrm{dn}\,u = -m\;\mathrm{sn}\,u\;\mathrm{cn}\,u. \tag{E.15}$$

Near $m = 0$ and $m = 1$ we have the following approximations for the Jacobi elliptic functions,

$$\mathrm{sn}\,u = \sin u + \mathcal{O}(m), \quad \mathrm{sn}\,u = \tanh u + \mathcal{O}(1-m), \tag{E.16}$$

$$\mathrm{cn}\,u = \cos u + \mathcal{O}(m), \quad \mathrm{cn}\,u = \ \mathrm{sech}\,u + \mathcal{O}(1-m), \tag{E.17}$$

$$\mathrm{dn}\,u = 1 + \mathcal{O}(m), \quad \mathrm{dn}\,u = \ \mathrm{sech}\,u + \mathcal{O}(1-m). \tag{E.18}$$

The complete elliptic integral of the second kind is simply written as E,

$$E = \int_0^{\frac{\pi}{2}} d\theta\,\left(1 - k^2\sin^2\theta\right)^{\frac{1}{2}}. \tag{E.19}$$

Differentiating with respect to k gives

$$\frac{dE}{dk} = \frac{1}{k}\,(E - K). \tag{E.20}$$

Similarly, if we differentiate K we find

$$\frac{dK}{dk} = -\frac{K}{k} + \frac{1}{k}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\left(1 - k^2 \sin^2\theta\right)^{\frac{3}{2}}}, \tag{E.21}$$

but,

$$\int_0^{\frac{\pi}{2}} d\theta \, \frac{d}{d\theta} \frac{k^2 \sin\theta\cos\theta}{\left(1 - k^2\sin^2\theta\right)^{\frac{1}{2}}} = E - (1 - k^2)\int_0^{\frac{\pi}{2}} \frac{d\theta}{\left(1 - k^2\sin^2\theta\right)^{\frac{3}{2}}} = 0, \tag{E.22}$$

so we have

$$\frac{dK}{dk} = \frac{1}{k}\left(\frac{E}{1 - k^2} - K\right). \tag{E.23}$$

Bibliography

[1] N. M. Davies, T. J. Hollowood, V. V. Khoze, and M. P. Mattis, *Gluino condensate and magnetic monopoles in supersymmetric gluodynamics*, Nucl. Phys. **B559** (1999) 123, [hep-th/9905015].

[2] N. M. Davies and V. V. Khoze, *On Affleck-Dine-Seiberg superpotential and magnetic monopoles in supersymmetric qcd*, JHEP **01** (2000) 015, [hep-th/9911112].

[3] N. M. Davies, T. J. Hollowood, and V. V. Khoze, *Monopoles, affine algebras and the gluino condensate*, hep-th/0006011.

[4] R. Rajaraman, *Solitons and instantons: An introduction to solitons and instantons in quantum field theory.* North Holland, 1982.

[5] A. I. Vainshtein, V. I. Zakharov, V. A. Novikov, and M. A. Shifman, *ABC of instantons*, Sov. Phys. Usp. **25** (1982) 195.

[6] S. Coleman, *Aspects of symmetry: Selected Erice lectures.* Cambridge University Press, 1985.

[7] G. 't Hooft, *Computation of the quantum effects due to a four- dimensional pseudoparticle*, Phys. Rev. **D14** (1976) 3432–3450.

G. 't Hooft, *Computation of the quantum effects due to a four- dimensional pseudoparticle: Erratum*, Phys. Rev. **D18** (1978) 2199.

[8] J. L. Gervais and B. Sakita, *Extended particles in quantum field theories*, Phys. Rev. **D11** (1975) 2943.

E. Tomboulis, *Canonical quantization of nonlinear waves*, Phys. Rev. **D12** (1975) 1678.

[9] F. A. Berezin, *The Method of Second Quantization.* Academic Press, New York, 1966.

[10] A. Borel, *Topology of Lie groups and characteristic classes*, Bull. Am. Math. Soc. **61** (1955) 397.

R. Bott, *On torsion in Lie groups*, Proc. Nat. Acad. Sci. **40** (1954) 586–588.

R. Bott, *An application of the Morse theory to the topology of Lie groups*, Bull. Soc. Math. France **84** (1956) 251.

[11] A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Y. S. Tyupkin, *Pseudoparticle solutions of the Yang-Mills equations*, Phys. Lett. **B59** (1975) 85.

[12] R. Jackiw and C. Rebbi, *Conformal properties of a Yang-Mills pseudoparticle*, Phys. Rev. **D14** (1976) 517.

[13] C. W. Bernard, N. H. Christ, A. H. Guth, and E. J. Weinberg, *Pseudoparticle parameters for arbitrary gauge groups*, Phys. Rev. **D16** (1977) 2967.

[14] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld, and Y. I. Manin, *Construction of instantons*, Phys. Lett. **A65** (1978) 185.

[15] M. F. Atiyah and I. M. Singer, *The index of elliptic operators: I*, Annals Math. **87** (1968) 484–530.

[16] E. F. Corrigan, D. B. Fairlie, S. Templeton, and P. Goddard, *A green's function for the general selfdual gauge field*, Nucl. Phys. **B140** (1978) 31.

[17] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, M. B. Voloshin, and V. I. Zakharov, *Supersymmetry transformations of instantons*, Nucl. Phys. **B229** (1983) 394.

[18] S. Coleman and J. Mandula, *All possible symmetries of the S matrix*, Phys. Rev. **159** (1967) 1251.

[19] R. Haag, J. T. Łopuszański, and M. Sohnius, *All possible generators of supersymmetries of the S matrix*, Nucl. Phys. **B88** (1975) 257.

[20] J. Wess and J. Bagger, *Supersymmetry and Supergravity.* Princeton University Press, 2nd ed., 1992.

[21] C. Bernard, *Gauge zero modes, instanton determinants, and quantum-chromodynamic calculations*, Phys. Rev. **D19** (1979) 3013.

[22] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, *Exact Gell-Mann-Low function of supersymmetric Yang-Mills theories from instanton calculus*, Nucl. Phys. **B229** (1983) 381.

[23] N. Dorey, T. J. Hollowood, V. V. Khoze, M. P. Mattis, and S. Vandoren, *Multi-instantons and Maldacena's conjecture*, JHEP **06** (1999) 023, [hep-th/9810243].

[24] N. Dorey, T. J. Hollowood, V. V. Khoze, M. P. Mattis, and S. Vandoren, *Multi-instanton calculus and the AdS/CFT correspondence in $\mathcal{N} = 4$ superconformal field theory*, Nucl. Phys. **B552** (1999) 88, [hep-th/9901128].

[25] J. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998) 231–252, [hep-th/9711200].

[26] N. Seiberg and E. Witten, *Electric - magnetic duality, monopole condensation, and confinement in $\mathcal{N} = 2$ supersymmetric Yang-Mills theory*, Nucl. Phys. **B426** (1994) 19–52, [hep-th/9407087].

N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation, and confinement in $\mathcal{N} = 2$ supersymmetric Yang-Mills theory: Erratum*, Nucl. Phys. **B430** (1994) 485–486.

[27] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $\mathcal{N} = 2$ supersymmetric QCD*, Nucl. Phys. **B431** (1994) 484–550, [hep-th/9408099].

[28] N. Dorey, V. V. Khoze, and M. P. Mattis, *Multi-instanton calculus in $\mathcal{N} = 2$ supersymmetric gauge theory*, Phys. Rev. **D54** (1996) 2921–2943, [hep-th/9603136].

[29] V. V. Khoze, M. P. Mattis, and M. J. Slater, *The instanton hunter's guide to supersymmetric $SU(N)$ gauge theory*, Nucl. Phys. **B536** (1998) 69, [hep-th/9804009].

[30] A. D'Adda and P. Di Vecchia, *Supersymmetry and instantons*, Phys. Lett. **B73** (1978) 162.

[31] D. Amati, K. Konishi, Y. Meurice, G. C. Rossi, and G. Veneziano, *Nonperturbative aspects in supersymmetric gauge theories*, Phys. Rept. **162** (1988) 169.

[32] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, *Instanton effects in supersymmetric theories*, Nucl. Phys. **B229** (1983) 407.

[33] J. Fuchs and M. G. Schmidt, *Instanton induced Green functions in the superfield formalism*, Z. Phys. **C30** (1986) 161.

[34] J. Fuchs, *A supersymmetric instanton calculus for $SU(N)$ gauge theories*, Nucl. Phys. **B272** (1986) 677.

[35] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, *Supersymmetric instanton calculus: Gauge theories with matter*, Nucl. Phys. **B260** (1985) 157–181.

[36] M. A. Shifman and A. I. Vainshtein, *On gluino condensation in supersymmetric gauge theories with $SU(N)$ and $O(N)$ groups*, Nucl. Phys. **B296** (1988) 445.

A. Y. Morozov, M. A. Olshanetsky, and M. A. Shifman, *Gluino condensate in supersymmetric gluodynamics (ii)*, Nucl. Phys. **B304** (1988) 291.

[37] D. Finnell and P. Pouliot, *Instanton calculations versus exact results in four-dimensional susy gauge theories*, Nucl. Phys. **B453** (1995) 225–239, [hep-th/9503115].

[38] A. Kovner and M. Shifman, *Chirally symmetric phase of supersymmetric gluodynamics*, Phys. Rev. **D56** (1997) 2396–2402, [hep-th/9702174].

[39] M. Shifman and A. Vainshtein, *Instantons versus supersymmetry: Fifteen years later*, hep-th/9902018.

[40] E. Witten, *Constraints on supersymmetry breaking*, Nucl. Phys. **B202** (1982) 253.

E. Witten, *Toroidal compactification without vector structure*, JHEP **02** (1998) 006, [hep-th/9712028].

[41] K. Konishi, *Anomalous supersymmetry transformation of some composite operators in SQCD*, Phys. Lett. **B135** (1984) 439.

[42] T. J. Hollowood, V. V. Khoze, W. Lee, and M. P. Mattis, *Breakdown of cluster decomposition in instanton calculations of the gluino condensate*, Nucl. Phys. **B570** (2000) 241, [hep-th/9904116].

[43] K. Intriligator and N. Seiberg, *Duality, monopoles, dyons, confinement and oblique confinement in supersymmetric $SO(N_c)$ gauge theories*, Nucl. Phys. **B444** (1995) 125–160, [hep-th/9503179].

K. Intriligator and P. Pouliot, *Exact superpotentials, quantum vacua and duality in supersymmetric $SP(N_c)$ gauge theories*, Phys. Lett. **B353** (1995) 471–476, [hep-th/9505006].

[44] C. G. Callan, Jr., R. Dashen, and D. J. Gross, *Toward a theory of the strong interactions*, Phys. Rev. **D17** (1978) 2717.

[45] A. A. Belavin, V. A. Fateev, A. S. Schwartz, and Y. S. Tyupkin, *Quantum fluctuations of multi-instanton solutions*, Phys. Lett. **B83** (1979) 317.

[46] V. A. Fateev, I. V. Frolov, and A. S. Schwartz, *Quantum fluctuations of instantons in the nonlinear sigma model*, Nucl. Phys. **B154** (1979) 1.

[47] A. M. Polyakov, *Quark confinement and topology of gauge groups*, Nucl. Phys. **B120** (1977) 429–458.

[48] N. Dorey, V. V. Khoze, M. P. Mattis, D. Tong, and S. Vandoren, *Instantons, three-dimensional gauge theory, and the Atiyah-Hitchin manifold*, Nucl. Phys. **B502** (1997) 59, [hep-th/9703228].

[49] D. J. Gross, R. D. Pisarski, and L. G. Yaffe, *QCD and instantons at finite temperature*, Rev. Mod. Phys. **53** (1981) 43.

[50] P. A. M. Dirac, *Quantized singularities in the electromagnetic field*, Proc. Roy. Soc. **A133** (1931) 60.

[51] G. 't Hooft, *Magnetic monopoles in unified gauge theories*, Nucl. Phys. **B79** (1974) 276–284.

[52] A. M. Polyakov, *Particle spectrum in quantum field theory*, JETP Lett. **20** (1974) 194–195.

[53] E. B. Bogomol'nyi, *Stability of classical solutions*, Sov. J. Nucl. Phys. **24** (1976) 449.

[54] M. K. Prasad and C. M. Sommerfield, *An exact classical solution for the 't Hooft monopole and the Julia-Zee dyon*, Phys. Rev. Lett. **35** (1975) 760–762.

[55] E. J. Weinberg, *Parameter counting for multi-monopole solutions*, Phys. Rev. **D20** (1979) 936–944.

[56] C. Callias, *Index theorems on open spaces*, Commun. Math. Phys. **62** (1978) 213.

[57] K. Lee and C. Lu, *$SU(2)$ calorons and magnetic monopoles*, Phys. Rev. **D58** (1998) 025011, [hep-th/9802108].

[58] B. J. Harrington and H. K. Shepard, *Periodic Euclidean solutions and the finite temperature Yang-Mills gas*, Phys. Rev. **D17** (1978) 2122.

[59] K. Lee, *Instantons and magnetic monopoles on $R^3 \times S^1$ with arbitrary simple gauge groups*, Phys. Lett. **B426** (1998) 323–328, [hep-th/9802012].

[60] T. C. Kraan and P. van Baal, *Periodic instantons with non-trivial holonomy*, Nucl. Phys. **B533** (1998) 627, [hep-th/9805168].

T. C. Kraan and P. van Baal, *Monopole constituents inside $SU(n)$ calorons*, Phys. Lett. **B435** (1998) 389, [hep-th/9806034].

[61] E. J. Weinberg, *Fundamental monopoles and multi-monopole solutions for arbitrary simple gauge groups*, Nucl. Phys. **B167** (1980) 500.

[62] E. Cohen and C. Gomez, *Chiral symmetry breaking in supersymmetric Yang-Mills theory*, Phys. Rev. Lett. **52** (1984) 237.

[63] G. 't Hooft, *Some twisted selfdual solutions for the Yang-Mills equations on a hypertorus*, Commun. Math. Phys. **81** (1981) 267.

[64] K. G. Wilson and J. Kogut, *The renormalization group and the ε expansion*, Phys. Rept. **12** (1974) 75–200.

M. A. Shifman and A. I. Vainshtein, *Solution of the anomaly puzzle in susy gauge theories and the Wilson operator expansion*, Nucl. Phys. **B277** (1986) 456.

M. A. Shifman and A. I. Vainshtein, *On holomorphic dependence and infrared effects in supersymmetric gauge theories*, Nucl. Phys. **B359** (1991) 571–580.

M. Dine and Y. Shirman, *Some explorations in holomorphy*, Phys. Rev. **D50** (1994) 5389–5397, [hep-th/9405155].

[65] M. E. Peskin, *Duality in supersymmetric Yang-Mills theory*, hep-th/9702094.

[66] S. Katz and C. Vafa, *Geometric engineering of $\mathcal{N}=1$ quantum field theories*, Nucl. Phys. **B497** (1997) 196–204, [hep-th/9611090].

C. Vafa, *On $\mathcal{N}=1$ Yang-Mills in four dimensions*, Adv. Theor. Math. Phys. **2** (1998) 497, [hep-th/9801139].

[67] O. Aharony, A. Hanany, K. Intriligator, N. Seiberg, and M. J. Strassler, *Aspects of $n=2$ supersymmetric gauge theories in three dimensions*, Nucl. Phys. **B499** (1997) 67, [hep-th/9703110].

[68] E. D'Hoker and D. H. Phong, *Lectures on supersymmetric Yang-Mills theory and integrable systems*, hep-th/9912271.

[69] P. Goddard and D. Olive, *Kac-Moody and Virasoro algebras in relation to quantum physics*, Int. J. Mod. Phys. **A1** (1986) 303.

[70] V. G. Kac, *Infinite Dimensional Lie Algebras.* Cambridge University Press, 3rd ed., 1990.

[71] N. Dorey, *An elliptic superpotential for softly broken $\mathcal{N} = 4$ supersymmetric Yang-Mills theory*, JHEP **07** (1999) 021, [hep-th/9906011].

[72] I. Affleck, M. Dine, and N. Seiberg, *Dynamical supersymmetry breaking in supersymmetric QCD*, Nucl. Phys. **B241** (1984) 493–534.

[73] K. Intriligator and N. Seiberg, *Lectures on supersymmetric gauge theories and electric- magnetic duality*, Nucl. Phys. Proc. Suppl. **45BC** (1996) 1–28, [hep-th/9509066].

[74] W. Nahm, *The construction of all self-dual multimonopoles by the ADHM method*, in *Monopoles in Quantum Field Theory: Proceedings of the Monopole Meeting, Trieste, Italy, December 1981* (N. S. Craigie, P. Goddard, and W. Nahm, eds.), p. 87, World Scientific, 1982.

W. Nahm, *All self-dual multimonopoles for arbitrary gauge groups*, in *Structural Elements in Particle Physics and Statistical Mechanics: Proceedings* (J. Honerkamp, K. Pohlmeyer, and H. Römer, eds.), vol. 82 of *NATO Advances Studies Institutes Series B: Physics*, p. 301, Plenum, 1983.

[75] E. Corrigan and P. Goddard, *Construction of instanton and monopole solutions and reciprocity*, Ann. Phys. **154** (1984) 253.

[76] N. J. Hitchin, *On the construction of monopoles*, Commun. Math. Phys. **89** (1983) 145.

[77] W. Nahm, *A simple formalism for the BPS monopole*, Phys. Lett. **B90** (1980) 413.

[78] H. Panagopoulos, *Multi - monopoles in arbitrary gauge groups and the complete $SU(2)$ two monopole system*, Phys. Rev. **D28** (1983) 380.

[79] S. A. Brown, H. Panagopoulos, and M. K. Prasad, *Two separated $SU(2)$ Yang-Mills-Higgs monopoles in the Atiyah-Drinfeld-Hitchin-Manin-Nahm construction*, Phys. Rev. **D26** (1982) 854.

[80] M. F. Atiyah and N. J. Hitchin, *The Geometry and Dynamics of Magnetic Monopoles.* Princeton University Press, 1988. M. B. Porter lectures.

[81] P. M. Sutcliffe, *BPS monopoles*, Int. J. Mod. Phys. **A12** (1997) 4663, [`hep-th/9707009`].

[82] G. W. Gibbons and N. S. Manton, *Classical and quantum dynamics of BPS monopoles*, Nucl. Phys. **B274** (1986) 183.

[83] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal field theory.* Springer, 1997. Chapter 13.

R. Gilmore, *Lie groups, Lie algebras and some of their applications.* Wiley, 1974.

H. Samelson, *Notes on Lie algebras.* Springer, 1990.

R. N. Cahn, *Semisimple Lie algebras and their representations.* Benjamin/Cummings, 1984.

[84] J. F. Adams, *Lectures on exceptional Lie groups.* University of Chicago Press, 1996.

[85] J. Baez, *This Week's Finds*, `http://math.ucr.edu/home/baez/TWF.html`.

[86] M. Abramowitz and I. A. Stegun, eds., *Handbook of Mathematical Functions.* Dover, 1965.

E. H. Neville, *Jacobian Elliptic Functions.* Oxford University Press, 1944.

E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis.* Cambridge University Press, 4th ed., 1927.

T. MacRobert, *Functions of a Complex Variable.* Macmillan, 1917.